http://ansinet.com/itj

ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 12 (18): 4705-4709, 2013
ISSN 1812-5638 / DOI: 10.3923/itj.2013.4705.4709

Ternary Perfect Sequences with a Few Zero Elements

Li Xu-Dong, He Ming-Xing and Wang Jing
School of Mathematics and Computer Engineering, Xihua University, Chengdu, 610039, Sichuan, China

Abstract: Ternary perfect sequences which include the binary perfect sequences as special cases, are introduced. The new notions of elementary transformations on ternary perfect sequences are brought forward. Necessary conditions for ternary perfect sequences with k zero elements are proposed. It is proved that there exist no ternary perfect sequences of odd lengths with one zero element and no ternary perfect sequences of odd lengths with two zero elements. It is also proved that there exist no ternary perfect sequences of any length with n adjacent zero elements and n is a positive integer which is greater than 1. An efficient search algorithm for ternary perfect sequences is given.

Key words: Ternary perfect sequences, elementary operations, periodic autocorrelation functions, energy efficiency

INTRODUCTION

Discrete-time sequences are called perfect sequences, if all the out-of-phase values of their periodic autocorrelation functions (PACF) are equal to zero. Such sequences are widely used for estimation of impulse-responses, detection of reflected waves, time synchronization, system identification and the synthesis of orthogonal matrices for source coding and so on (Fan and Darnell, 1996). It is desirable to use binary sequences over the alphabet set {±1}. Unfortunately, the only binary perfect sequence known to exist is of length 4, i.e., {1, 1, 1, -1} (not counting its equivalent forms {-1, -1, -1, 1}, {-1, 1, 1, 1} and {1, -1, -1, -1}, etc.) and it has been conjectured that no other binary perfect sequences exist [1]. Binary perfect sequences of odd lengths do not exist since all values of their autocorrelation function must necessarily be odd integers (Helleseth and Kumar, 1998). If the sequence elements are nonbinary, real- or complex-valued, then many perfect sequences can be synthesized. Multilevel perfect sequences over integers have been considered and four-level perfect sequences of lengths 12 and 28 over the alphabet set {±1, ±2} have been found (Li *et al*., 2011). Ternary perfect sequences over the alphabet {0, ±1} have been studied by Chang (1967), Moharir (1977), Shedd and Sarwate (1979), Ipatov (1979, 1980), Hoholdt and Justesen (1983), Luke (1988), Antweiler *et al.* (1990) and Boztas *et al.* (2010). Ternary perfect sequences of lengths $(3^n-1)/2$, where n is odd, have been constructed by Chang using polynomials over GF(3). Some combinatorial conditions necessary for the existence of the ternary perfect sequences are discussed by Moharir. Systematic methods for synthesizing ternary perfect sequences of length $(q^{2r+1}-1)/(q-1)$ have been presented, where q is some power of a prime number (denoted by p) and r is an integer. For p = 2, a construction method based on the Singer difference sets has been given by Hoholdt and Justesen. For p ≠ 2, Ipatov constructed a class of ternary perfect sequences by mapping the elements of a shift register sequence over $GF(p^s)$. Nonbinary "extended" PSK sequences with the perfect autocorrelation property which include ternary perfect sequences as special cases, are designed by Boztas and Parampalli. In general, it is desirable that the number of zeros in the ternary perfect sequences is as small as possible. Ternary perfect sequences with a small number of zero elements are investigated in this study.

The concept of perfect sequences is introduced in section 2. Properties of ternary perfect sequences are given in section 3. Necessary conditions for ternary perfect sequences with k zero elements are brought forward in section 4. The existence of ternary perfect sequences with one zero element is discussed in section 5 and those with two zero elements in section 6. Concluding remarks are given in section 7.

CONCEPT OF PERFECT SEQUENCES

Let $a = (a_0, a_1, \ldots, a_{N-1})$ be a real-valued sequence of length (or period) N, the PACF of the sequence a is defined as:

$$R_a = \sum_{i=0}^{N-1} a_i a_{i+\tau}, \tau = 0, 1, 2, \cdots, N-1 \quad (1)$$

Corresponding Author: Li Xu-Dong, School of Mathematics and Computer Engineering, Xihua University, Chengdu, 610039, Sichuan, China

where, the subscript is computed modulo N. Similarly, the periodic cross-correlation function of the sequence a and $b = (b_0, b_1, \ldots, b_{N-1})$ is given by:

$$R_{ab}(\tau) = \sum_{i=0}^{N-1} a_i b_{i+\tau}, \quad \tau = 0, 1, 2, \cdots, N-1 \tag{2}$$

The sequence $a = (a_0, a_1, \ldots, a_{N-1})$ is said to be perfect if the PACF is given by:

$$R_a(\tau) = \begin{cases} E_a, & \tau = 0, \\ 0, & \tau \neq 0. \end{cases} \tag{3}$$

where, the energy E_a of the sequence a is given by:

$$E_a = a_0^2 + a_1^2 + \cdots + a_{N-1}^2 \tag{4}$$

For most applications, the sequence a should possess a high energy efficiency η_a defined by:

$$\eta_a = \frac{\sum_{i=0}^{N-1} |a_n|^2}{N \cdot \max\{|a_n|\}} \tag{5}$$

or, equivalently, a low peak factor $1/\eta_a$. This parameter is of particular importance for sequences whose element amplitudes are not constant. The energy efficiency of any polyphase sequence is equal to 1 which is the highest possible value. For a ternary perfect sequence of a fixed length, the smaller the number of zero elements, the higher the energy efficiency.

PROPERTIES OF TERNARY PERFECT SEQUENCES

There are some transformations of ternary sequences which preserve the perfect property of PACF. Theorem 1 summarizes such transformations.

Theorem 1: Fan and Darnell (1996): Let $a = (a_0, a_1, \ldots, a_{N-1})$ be a ternary perfect sequence of length N. The following transformations preserve the perfect periodic autocorrelation function of the original sequence and are respectively defined as:

$$a_n \rightarrow a_{N-n-1} \ (n = 0, 1, 2, \cdots, N-1) \tag{6}$$

$$a_n \rightarrow -a_n, \ (n = 0, 1, 2, \cdots, N-1) \tag{7}$$

$$a_n \rightarrow a_{n+m}, \ (n = 0, 1, 2, \cdots, N-1) \tag{8}$$

where, m is any integer and the subscript is computed modulo N. Furthermore, the transformed sequences resulted from Eq. 6-8 are denoted by $\bar{a}$, a_ and $T^m a$, respectively, where T denotes the operator that shifts the sequence cyclically to the left by one place.

We call the above transformations the elementary transformations for ternary perfect sequences. When a ternary perfect sequence can be obtained from another via some successive applications of the elementary transformations, the two sequences are said to be equivalent. The set of all sequences which are equivalent to a sequence a form an equivalent class and is denoted by [a]. This equivalence relationship induces a partition of the set of all ternary perfect sequences of same length. Any ternary perfect sequence chosen from an equivalence class is defined as the representative of the equivalence class and we also call it a representative ternary perfect sequence hereafter. So the set of inequivalent representative ternary perfect sequences can represent the set of all ternary perfect sequences of some fixed length.

Theorem 2: Fan and Darnell (1996): Let a and b be two ternary perfect sequences of respective lengths N_1 and N_2 which are relatively prime. By repeating the sequence a N_2 times and the sequence b N_1 times and multiplying them elementwise, the resultant product sequence forms a ternary perfect sequence of length N_1N_2.

Theorems 1 and 2 can be generalized to any perfect sequences.

NECESSARY CONDITIONS FOR TERNARY PERFECT SEQUENCES WITH *K* ZERO ELEMENTS

For any ternary perfect sequence $a = (a_0, a_1, \ldots, a_{N-1})$ with elements in {-1, 0, +1}, if there are m_+ +1s, m -1 sec and m_0 0 sec such that the total number of digits $N = m_+ + m\_ + m_0$, then the following relationship can be shown to hold [1]:

$$(m_+ + m\_)^2 = m_+ m\_ \tag{9}$$

Without loss of generality, we can assume $m_+ \geq m\_$.

By applying Eq. 9 and the properties of ternary perfect sequences, we have the following Lemma.

Lemma 3: Let $a = (a_0, a_1, \ldots, a_{N-1})$ be a ternary perfect sequence of length N with m_0 0 sec, m_+ +1 sec and m_ -1 sec. We have:

$$N = k^2 + m_0 \tag{10a}$$

Table 1: Parameters of ternary sequences of length less than 60

N	m_+	$m\_$	m_0	η	Representative ternary perfect sequences
6	3	1	2	0.6667	(1, 0, -1, 1, 0, 1)
13	6	3	4	0.6923	(1, 0, 1, 0, 0, 1, -1, 1, 0, -1, -1, 1, 1)
21	10	6	5	0.7619	(1, 1, 1, 1, 1, -1, 1, 0, 1, 0, -1, 1, 1, -1, 0, 0, 1, -1, 0, -1, -1)
26	6	3	17	0.3462	(1, 1, 1, -1, 0, 1, 0, 0, 1, -1, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)
31	10	6	15	0.5161	(1, 0, 1, 1, 0, 0, 1, 1, -1, 1, -1, 0, 0, 0, 1, 1, 0, 1, -1, -1, 0, 1, 0, -1, 0, 0, 0, 0, -1, 0, 0)
31	15	10	6	0.8065	(1, 0, 1, 1, 1, -1, 0, -1, -1, -1, 0, 1, -1, 1, 1, -1, 0, 1, -1, 1, -1, 1, 1, 1, -1, -1, 1, 1, 1, 0, 0)
33	15	10	8	0.7576	(1, 1, 1, 1, 0, 1, 1, -1, 1, -1, 1, 0, 1, -1, 0, 0, -1, 0, -1, -1, 1, 1, 1, 0, -1, 1, 0, -1, 1, 1, -1, -1, 0)
57	28	21	8	0.8596	(1, 1, 1, 1, 1, -1, 1, 0, -1, 1, -1, 0, -1, -1, -1, 1, -1, -1, 0, 1, -1, 1, 1, -1, -1, 1, 1, 0, -1, 1, -1, 1, 0, 0, 1, 0, 1, 1, 1, 1, -1, -1, -1, 1, 1, 0, 1, 1, 1, -1, -1, -1, 1, 1, -1, 1, -1)

$$m_+ = \frac{k(k+1)}{2} \tag{10b}$$

$$m\_ = \frac{k(k-1)}{2} \tag{10c}$$

$$N = k^2+m_0$$

where, k is a positive integer.

An efficient search algorithm for ternary perfect sequences is given as follows.

Input: Parameters of ternary perfect sequences, i.e. length N and the number of zero elements m_0 of the ternary perfect sequences:

- Determine m_+ and m_- by using Theorem 4
- Generate all ternary sequences of length N with m_0 0 sec, m_+ +1 sec and $m\_$ -1 sec
- Compute the PACF of the above ternary sequences
- Verify if the computed PACF of the above ternary sequences satisfy Eq. 3 and if so, store them as ternary perfect sequences

Output: All ternary perfect sequences of length N with m_0 0 sec, m_+ +1 sec, $m\_$ -1 sec and energy efficiency η.

Based on examples from (Fan and Darnell, 1996; Chang, 1967; Hoholdt and Justesen, 1983; Antweiler *et al.*, 1990), representative ternary perfect sequences of lengths less than 60 are summarized in Table 1. The fifth and sixth rows in Table 1 show that ternary perfect sequences constructed by Moharir have lower energy efficiency than the others. As exemplified by Table 1, known ternary perfect sequences with one or two zero elements are scarce, so it is necessary to study the existence of such sequences.

EXISTENCE OF TERNARY PERFECT SEQUENCES WITH ONE ZERO ELEMENT

Theorem 4: Let a = (a_0, a_1,..., a_{N-1}) be a ternary perfect sequence of length N with one zero element. We have:

$$N = k^2+1 \tag{11}$$

where, k is a positive odd integer.

Proof: By Eq. 10a, we have $N = k^2+1$.

Since, a = (a_0, a_1,..., a_{N-1}) is a ternary perfect sequence of length N with a zero element, we have:

$$a_0a_1+a_1a_2+\ldots+a_{N-2}a_{N-1}+a_{N-1}a_0 = 0 \tag{12}$$

There are exactly two terms on the left hand side of Eq. 12 equal to 0. Then N-2 must be even, namely, N must be even and hence k must be a positive odd integer.

Theorem 4 shows that there exist no ternary perfect sequences of odd lengths with one zero element.

Using the efficient search algorithm for ternary perfect sequences, we have obtained the result concerning the non-existence of ternary perfect sequences of lengths less than 49 with a zero element. So it is conjectured that there exist no ternary perfect sequences of any lengths with one zero element.

EXISTENCE OF OF TERNARY PERFECT SEQUENCES WITH TWO ZERO ELEMENTS

According to the locations of the two zero elements, ternary perfect sequences with two zero elements can be classified into two cases:

- With two adjacent zeros
- With two non-adjacent zeros

The following theorem characterizes some properties of ternary perfect sequences with two zero elements.

Theorem 5: Let a = (a_0, a_1,..., a_{N-1}) be ternary perfect sequence of length N with two zero elements, then we have:

$$N = k^2+2 \tag{13}$$

where, k is a positive even integer and the two zeros are not adjacent in ternary perfect sequences.

Proof: In view of Eq. 10a, we have $N = k^2+2$:

- With two adjacent zeros

Since, $a = (a_0, a_1,\ldots, a_{N-1})$ is a ternary perfect sequence of length N with two zero elements, we have:

$$a_0a_1+a_1a_2+\ldots+a_{N-2}a_{N-1}+a_{N-2}a_0 = 0 \quad (14)$$

$$a_0a_1+a_1a_3+\ldots+a_{N-3}a_{N-1}+a_{N-2}a_0+a_{N-1}a_1 = 0 \quad (15)$$

There are three terms on the left hand side of Eq. 14 equal to 0. Then N-3 must be even, namely N must be odd. However, there are four terms on the left hand side of Eq. 15 equal to 0. Then N-4 must be even, namely N must be even. Obviously, a contradiction is resulted. Thus there exist no ternary perfect sequences of even length N with two adjacent zero elements.

- With two non-adjacent zeros

Since, $a = (a_0, a_1,\ldots, a_{N-1})$ is a ternary perfect sequence of length N with two zero elements, there are four terms on the left hand side of Eq. 14 equal to 0. Then N-4 must be even, namely, N must be even and hence k is a positive even integer.

Theorem 5 shows that there exist no ternary perfect sequences of odd lengths with two zero elements.

Using the efficient search algorithm for ternary perfect sequences, we have obtained that only one ternary perfect sequences of length 6 within ternary sequences of lengths less than 37 with two zero elements.

Lemma 6: Ipatov (1980): Ipatov ternary perfect sequences are a large class of ternary perfect sequences of length $N = (q^m-1)/(q-1)$ where m is an odd number and $q = p^s$, p is an odd prime and s is an integer.

It is obvious that $N = q^{m-1}+\ldots+q+1$ is an odd number. In view of Lemma 6 and Theorem 5, we have the following corollary.

Corollary 7: There exist no Ipatov ternary perfect sequences with two zero elements.

EXISTENCE OF OF TERNARY PERFECT SEQUENCES WITH *N* ADJACENT ZERO ELEMENTS

Theorem 7: There exist no ternary perfect sequences of lengths N with n zero elements which are adjacent, where n is a positive integer and $n\geq 2$.

Proof: Let $a = (a_0, a_1,\ldots, a_{N-1})$ be ternary perfect sequence of length N with n zero elements which are adjacent, then, we have:

$$a_0a_1+a_1a_2+\ldots+a_{N-2}a_{N-1}+a_{N-2}a_0 = 0 \quad (16)$$

$$a_0a_1+a_1a_3+\ldots+a_{N-3}a_{N-1}+a_{N-2}a_0+a_{N-1}a_0 = 0 \quad (17)$$

There are n+1 terms on the left hand side of Eq. 16 equal to 0. Then N-(n+1) must be even. However, there are n+2 terms on the left hand side of Eq. 17 equal to 0. Then N-(n+2) must be even. Obviously, a contradiction is resulted. Thus there exist no ternary perfect sequences of even length N with n zero elements which are adjacent, where n is a positive integer.

CONCLUSION

It Some properties of ternary perfect sequences are summarized. The new notions of elementary transformations on ternary perfect sequences are brought forward. Necessary conditions for ternary perfect sequences with k zero elements are proposed. It is proved that there exist no ternary perfect sequences of even lengths with one zero element and no ternary perfect sequences of odd lengths with two zero elements. It is also proved that there exist no ternary perfect sequences of any length with $n(n\geq 2)$ adjacent zero elements. Using the efficient search algorithm for ternary perfect sequences, we have obtained the result concerning the non-existence of ternary perfect sequences of lengths less than 49 with one zero element and only one ternary perfect sequences of length 6 within ternary sequences of lengths less than 37 with two zero elements. Constructing ternary perfect sequences of even lengths with two non-adjacent zero elements is an open problem.

ACKNOWLEDGMENTS

The authors would like to thauk for the support by Natural Science Fmmdation of China rmder the Grant 51275210 and 51204181. The authors also thauk for the support by Independent Research Fmmdation of Jiangnan University under the Grant JUSRP11210.

This work was supported by the National Science Foundation of China (NSFC, No. 60773035), the Key Natural Science Fund of Sichuan Education Department (Grant No. 13ZA0031), the Key Scientific research fund of Xihua University (Grant No. R1222628), Open Research Fund of Key Laboratory of Xihua University (Grant No. S2jj2011-020), Distinguished Young Fund of the School of Mathematics and Computer Engineering of Xihua University, the National Key Technology R&D program (No.2011BAH26B00), the International Cooperation Project of Sichuan Province (No. 2009HH0009) and the Fund of Key Disciplinary of Sichuan Province (No.SZD0802-09-1).

REFERENCES

Antweiler, M.F.M., L. Bomer and H.D. Luke, 1990. Perfect Ternary Arrays. IEEE Trans.Inform. Theor., 36: 696-705.

Boztas, S. and U. Parampalli, 2010. Nonbinary sequences with perfect and nearly perfect autocorrelations. Proceedings of the IEEE International Symposium Information Information Theory, June 13-18, 2010, Austin, TX, USA., pp:1300-1304.

Chang, J.A., 1967. Ternary sequence with zero correlation. Electron. Lett., 55: 1211-1213.

Fan, P.Z. and M. Darnell, 1996. Sequence Design for Communications Applications. Wiley, RSP, New York, pp:169-196.

Helleseth,T. and P. V. Kumar, 1998. Sequences With Low Correlation. In: Handbook in Coding Theory, Pless V.S. and W. C. Huffman, (Eds.)., Vol. 2, Chapter 21, Amsterdam: Elsevier Science B.V., Netherland, pp:1765-1853.

Hoholdt, T. and , J. Justesen, 1983. Ternary sequences with perfect periodic autocorrelation. IEEE Trans. Inform. Theor., 29: 597-600.

Ipatov, V.P., 1979. Ternary sequences with ideal autocorrelation properties. Radio Eng. Electron. Phys., 24: 75-79.

Ipatov, V.P., 1980. Contribution to the theory of sequences with perfect periodic autocorrelation properties. Radio Eng. Electron. Phys., 25: 31-34.

Li, X.D., P.X. Fan, W.H. Mow and M. Darnell, 2011. Multilevel Perfect Sequences over Integers. Electron. Lett., 47: 496-497.

Luke, H.D., 1988. Sequences and arrays with perfect periodic correlation. IEEE Truns. Aerosp. Elrctron. Syst., 24: 287-294.

Moharir, P.S., 1977. Generalized PN sequences (Corresp.). IEEE Trans. Inform. Theor., 23: 782-784.

Shedd, D.A. and , D.V. Sarwate, 1979. Construction of sequences with good correlation properties (Corresp.). IEEE Trans. Inform. Theor., 25: 94-97.