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Animproved Block Diagonal Preconditioners for Non-symmetric Indefinite Linear Systems

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Abstract: Based on the block diagonal preconditioners by Cao in the study [Zhi-Hao Cao, A note on block diagonal and constraint preconditioners for non-symmetric indefinite linear systems, International Journal of Computer Mathematics, 83(4) (2006):383-395], we present a new block diagonal preconditioners for non-symmetric indefinite linear system. Moreover, we analyses the properties of the corresponding preconditioned matrices.

Key words: Saddle point systems, preconditioners, eigenvalues

INTRODUCTION

We consider the block 2×2 linear systems of the form:

$$\bar{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

where, $A\in R^{n\times n}$ is non-singular and B and $C\in R^{n\times n}(m<n)$ are of full rank, frequently appears in the solution of generalized saddle point problems Eq. 1 which include the linearized Navier-Stokes equations and so forth.

Let:

$$A = G-E \tag{2}$$

be a splitting of A, where $G\in R^{n\times n}$ is non-singular. De Sturler and Liesen (2005) constructed a detailed analysis for block diagonal preconditioners deriving from the (1,1)-block of the matrix $\bar{A}$ in Eq. 1 and the block diagonal (Saad, 1996). Preconditioner is defined as follows:

$$\bar{G}_+ = \begin{pmatrix} G & 0 \\ 0 & CG^{-1}B^T \end{pmatrix} \tag{3}$$

Murphy *et al.* (2000) proposed the following block diagonal preconditioner:

$$\bar{G}_{+,0} = \begin{pmatrix} A & 0 \\ 0 & CA^{-1}B^T \end{pmatrix} \tag{4}$$

Cao (2006) presented the following block diagonal preconditioner:

$$\bar{G}_- = \begin{pmatrix} G & 0 \\ 0 & CG^{-1}B^T \end{pmatrix} \tag{5}$$

In particular, when G = A in Eq. 4, the diagonal preconditioner is denoted by $\bar{G}_{-,0}$, i.e.:

$$\bar{G}_{-,0} = \begin{pmatrix} A & 0 \\ 0 & -CA^{-1}B^T \end{pmatrix} \tag{6}$$

and the corresponding preconditioned matrix $\bar{G}_{-,0}\bar{A}$ is diagonalizable and has at most three distinct eigenvalue:

$$1, \frac{1+\sqrt{3}i}{2}$$

Other authors further studied generalized saddle point problems Eq. 1, please refer to the literature (Zhang *et al.*, 2009, 2010, 2011, 2012; Huang and Zhang, 2009; Zhang and Cheng, 2013).

In this study, based on the block diagonal prenconditioner constructed by Cao (2006), we consider

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the block diagonal preconditioner (α<0) which is defined as follows:

$$\bar{G} = \begin{pmatrix} \alpha G & 0 \\ 0 & -\frac{1}{\alpha} CG^{-1}B^T \end{pmatrix} \quad (7)$$

In particular, when G = A in Eq. 7, the diagonal preconditioner is denoted by $\bar{G}_0$, i.e.:

$$\bar{G}_0 = \begin{pmatrix} \alpha A & 0 \\ 0 & -\frac{1}{\alpha} CA^{-1}B^T \end{pmatrix} \quad (8)$$

We will prove that the corresponding preconditioner matrix $\bar{G}_0\bar{A}$ is diagonlizable and has at most three distinct eigenvalues:

$$\frac{1}{\alpha}, \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

Remark 1: When α = 0, the block preconditions $\bar{G}$ and $\bar{G}_0$ reduce to the block preconditioners $\bar{G}_-$ and $\bar{G}_{-,0}$ (Cao, 2006), respectively. So, the block preconditioners considered in this study is a generalization of (Cao, 2006).

SPECTRAL ANALYSIS

Based on the new block preconditioned matrix $\bar{G}^{-1}\bar{A}$, similar to the analysis of (Cao, 2006), we give the following result to describe the spectral distribution:

$$\bar{G}^{-1}\bar{A} = \begin{pmatrix} \frac{1}{\alpha} G^{-1}A & \frac{1}{\alpha} G^{-1}B^T \\ -\frac{1}{\alpha}(CG^{-1}B^T)^{-1}C & 0 \end{pmatrix} \quad (9)$$

Let $G^{-1}E = T = I - G^{-1}A$ is the iterative matrix induced by the splitting Eq. 2 of A. Define $M = (CG^{-1}B^T)^{-1}C$ and $N = G^{-1}B^T$, then we may obtain:

$$\bar{F}(T) = \bar{G}^{-1}A = \begin{pmatrix} \frac{1}{\alpha}(I - T) & \frac{1}{\alpha} N \\ -\frac{1}{\alpha} M & 0 \end{pmatrix} \quad (10)$$

where, $M \times R^{m\times n}$, $N \in R^{n\times m}$ and $(NM)^2 = NM$.

Let $U^1 = [u_1,\ldots, u_{n-m}] \in R^{n\times(n-m)}$ form a basis of N(NM) the null space of NM and let $U_2 = [u_{n-m+1},\ldots, u_n] \in R^{n\times m}$ form a basis of R(NM) the range of NM, Thus, $[u_1, u_2] \in R^{n\times n}$ is non-singular and:

$$NM[U_1, U_2] = [U_1, U_2]\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad (11)$$

Theorem 1: The block diagonal preconditioned matrix $\bar{F}(0) = \bar{G}_0^{-1}\bar{A}$ is diagonalizable and has three distinct eigenvalues:

$$\frac{1}{\alpha}, \lambda_\pm = \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

Proof: Denote $\bar{T} = \bar{F}(0)$, then from Eq. 10 we obtain:

$$\bar{T} = \begin{pmatrix} \frac{1}{\alpha} I_n & \frac{1}{\alpha} N \\ -\frac{1}{\alpha} M & 0 \end{pmatrix}$$

and:

$$\bar{T}^2 = \begin{pmatrix} \frac{1}{\alpha^2}(I_n - NM) & \frac{1}{\alpha^2} N \\ -\frac{1}{\alpha^2} M & -\frac{1}{\alpha^2} I_m \end{pmatrix}$$

So:

$$\bar{T}^2 - \frac{1}{\alpha}\bar{T} = \begin{pmatrix} -\frac{1}{\alpha^2} NM & 0 \\ 0 & -\frac{1}{\alpha^2} I_m \end{pmatrix}$$

Then we have:

$$(\bar{T}^2 - \frac{1}{\alpha}\bar{T})^2 = -\frac{1}{\alpha^2}(\bar{T}^2 - \frac{1}{\alpha}\bar{T})$$

$$\Leftrightarrow \bar{T}(\bar{T} - \frac{1}{\alpha} I)(\bar{T}^2 - \frac{1}{\alpha}\bar{T} + \frac{1}{\alpha^2}) = 0$$

$$\Leftrightarrow \lambda(\lambda - \frac{1}{\alpha})(\lambda^2 - \frac{1}{\alpha}\lambda + \frac{1}{\alpha^2}) = 0$$

which have four distinct roots:

$$0, \frac{1}{\alpha}, \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

Since, $\bar{F}(0)$ is non-singular, it has at most three distinct eigenvalues:

$$\frac{1}{\alpha}, \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

Let $v = [x^T, y^T]^T$ be an eigenvector corresponding to the eigenvalue 1, then we have:

$$\frac{1}{\alpha}x + \frac{1}{\alpha}Ny = \frac{1}{\alpha}x, -\frac{1}{\alpha}Mx = y$$

which implies:

$$N_y = 0,\ NMx = 0,\ \text{i.e., } x \in N(NM),\ y = 0$$

Thus, we obtain that the eigenvector matrix corresponding to the eigenvalue 1 is:

$$\begin{pmatrix} U_1 \\ 0 \end{pmatrix} \in R^{n\times(n-m)}$$

Let $y = [x^T, y^T]^T$ be the eigenvector corresponding to the eigenvalue:

$$\lambda_\pm = \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

Then we can obtain:

$$(i)\frac{1}{\alpha}x + \frac{1}{\alpha}Ny = \lambda_\pm x, ii) - \frac{1}{\alpha}Mx = \lambda_\pm y$$

From (ii) we have:

$$y = -\frac{1}{\lambda_\pm \alpha}Mx$$

Inserting this into (i) and multiplying the resulting equation by $\lambda_\pm$ yields. Since:

$$\lambda_\pm^2 - \frac{1}{\alpha}\lambda_\pm + \frac{1}{\alpha^2} = 0$$

we have:

$$NMx = (\frac{1}{\alpha} - \lambda_\pm)\lambda_\pm \alpha^2 x = -\alpha^2(\lambda_\pm^2 - \frac{1}{\alpha}\lambda_\pm)x = -\alpha^2\frac{1}{\alpha^2} = x,$$

i.e., $x \in R(NM)$. Thus, we obtain the eigenvector matrices:

$$\begin{pmatrix} U_2 \\ -\lambda_+^{-1}MU_2 \end{pmatrix}, \begin{pmatrix} U_2 \\ -\lambda_-^{-1}MU_2 \end{pmatrix}$$

corresponding to the eigenvalue:

$$\lambda_+ = \frac{1}{\alpha}\frac{1 + \sqrt{3}i}{2}$$

and the eigenvalue:

$$\lambda_- = \frac{1}{\alpha}\frac{1 - \sqrt{3}i}{2}$$

respectively. Therefore, the eigienvector matrix Y(0) of $\bar{F}(0)$ is given by:

$$Y(0) = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & -\lambda_+^{-1}MU_2 & -\lambda_-^{-1}MU_2 \end{pmatrix}$$

Remark 2: When α increases, the three distinct eigenvalues:

$$\frac{1}{\alpha}, \frac{1}{\alpha}\frac{1 \pm \sqrt{3}i}{2}$$

of the preconditioned matrix $\bar{G}_0^{-1}\bar{A}$ are strongly clustered.

CONCLUSION

In this study, based on the block diagonal preconditioners presented by De Sturler and Liesen (2005), Murphy *et al*. (2000) and Cao (2006), we consider the new block diagonal preconditioners applied to the problems of solving non-symmetric indefinite linear systems. Theoretical analysis shows that the eigenvalues of the preconditioned matrix $\bar{G}_0$ considered in this study are strongly clustered.

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