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Trust Region Method for Network Design Problem with Equilibrium Constrains

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Abstract: The continuous design problem with equilibrium constrains is addressed in this study which is illustrated by a bi-level model. The upper-level problem is a mathematical program generally to minimize the total system cost, at the lower level the network users make choices with regard to route conditions following the user equilibrium principle. The lower is an equilibrium assignment model stated by variational inequality. A solution algorithm based on sensitivity analysis is designed for the model proposed. Finally, a numerical example is given to illustrate the application of the model and algorithm and some conclusions are drawn.

Key words: Network design, sensitivity analysis, trust-region, variational inequality

IINTRODUCTION

Modern urban road transportation network design is to optimize the system performance, make the traveler arrive in destination conveniently, quickly, safely and comfortably. The Network Design Problem (NDP) involves the optimal decision on the expansion of a street and highway system in response to a growing demand for travel. It has emerged as an important area for progress in handling effective transport planning, because the demand for travel on the roads is growing fast, while resources available for expanding the system capacity remain limited. Historically, this problem has been roughly classified into two different forms: a discrete form dealing with the additions of new links to an existing road network (DNDP) and a continuous form dealing with the optimal capacity expansion of existing links (CNDP). In whichever form, the objective of NDP is to optimize a given system performance measure such as to minimize total system travel cost, while accounting for the route choice behavior of network users (Yang and Bell, 1998). The decisions made by road planners in?uence the route choice behavior of network users which is normally described by a net-work user equilibrium model.

Mathematically, the bi-level programming is a good technique to describe this hierarchical property of the NDP with an equilibrium constraints. Generally the upper level problem is to minimize the total system cost and the lower level problem is to characterize the UE trafic flow pattern (Zhang *et al*., 2002).

Equilibrium models are often used in transportation network analysis, where each user is assigned to the least cost path between his origin and destination in a user-optimal manner. The result of the assignment is an estimate of traffic volume on each link in the network and the associated measures of system performance. A general problem in transportation system analysis and one that has more sound behavioral foundation, is the traffic assignment problem with elastic demand where the demand between every Origin-Destination (O-D) pair is assumed to be influenced by the level of service on the network. Friesz *et al*. (1990) developed a sensitivity analysis method for the general network equilibrium problem formulated as variational inequalities. An efficient approach has also been proposed to calculate the derivatives of the equilibrium link flows with respect to perturbation parameters in both the performance function and travel demand (Conn *et al*., 2000).

The sensitivity analysis methods are designed to calculate the derivatives of decision variables and constraint multipliers with respect to a variety of perturbation parameters. This derivative information could allow one to calculate a nearby equilibrium solution resulting from a variety of parameter perturbations once an equilibrium solution has been obtained.

A large number of scholars have investigated the NDP in one way or another over the past two decades (Wang *et al*., 2013). But they are largely considered network design of symmetric transportation equilibrium assignment problem which the upper model can be stated as optimization problem. While the asymmetric transportation assignment can not be stated optimization problem, thus the network design problem is stated by a bi-level model with equilibrium constraints.

BI-LEVEL NETWORK DESIGN PROBLEM FOR CNDP

Basic Bi-level programming model: The bi-level programming problem can be defined as follows:

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$$\text{P1: (U1)} \min_x F(x,y)$$

s.t.:

$$G(x,y) \leq 0$$

where, y = y (x) is implicitly defined by:

$$\text{(L1)} \min_y f(x,y)$$

s.t.:

$$G(x,y) \leq 0$$

Obviously, the bi-level programming model consists of two submodels, (U1) which defined as an upper level problem and (L1) which is a lower level problem. F and X are the objective function and decision vectors of upper level decision-makers or system managers, G and g are the constraint sets of the upper level and lower level decision vectors. F and y are the objective function and decision vectors of lower level decision-makers. y = y (x) is usually called the reaction or response function.

Bi-level programming models have a fairly long list of applications, especially in the economic discipline. For example, they have been applied to natural resource management, project selection, strategic planning for the agriculture sector and regional development. Bi-level programming applications to network design problems can be classified into two categories. The first is formulation of the problem in a linear bi-level network design problem (Van der Weijde *et al*., 2013), the second category is formulation of the problem in a nonlinear problem (Dafermos and Nagurney, 1984).

Models: To state the problem, let α be an index for links in the network, A be the set of links, w be an index for O-D pairs, w be the set containing indices of all O-D pairs, R be the set of paths in the network, R be the set of paths between O-D pair w∈W, $x = (...., x_a, ...)^T$ be the link flow vector, where x_a denotes the flow on link α∈A, y_a be the capacity increase level for link a, $y = (..., y_a, ...)$, t (x, y) be the travel cost vector whose elemen t_a (x, y)

Denotes the travel cost for link α, the link cost function t (x, y) is assumed to be strictly monotone, $f = (..., f_r, ...)$ be the path flow vector, d be the demand vector. $\Delta = \delta_{ar}$ is the link/path incidence matrix where δ_{ar} is1 if path r uses link α and 0 otherwise; $\Lambda = \delta_{wr}$ is the O-D pair/path incidence matrix where δ_{wr} is 1 if $r \in R_w$ and 0 otherwise; $G_a(y_a)$ be the investment function for capacity increases,θ be coefficient converting construction cost to travel cost.

Model is formulated as follows:

P2:

$$\text{Min } Z = \sum_{a\in A} x_a t_a(x,y) + \theta \sum_{a\in A} G_a(y_a)$$

(U2):

$$\text{S.t.} y_a \geq 0, \forall a \in A$$

(L2):

$$t(x,y)^T(x'-x) \geq 0, \forall x' \in \Omega',$$

Where:

$$\Omega' = \{x | x = \Delta f, \Lambda f = d, f \geq 0\}. \quad (1)$$

Therefore,:

$$x = \Delta f$$

Let:

$$T(f,y) = \Delta^T t(\Delta f, y)$$

Then (L2) can be written as:

$$T(f,y)^T(f'-f) \geq 0, \forall f' \in \Omega,$$

Where:

$$\Omega = \{f | \Lambda f = d, f \geq 0\}.$$

The lower-level model is a Variational Iquality (VI), in the bi-level model described above, the objective function of upper-level model represents the total cost; the lower-level model is an equilibrium assignment model. Constraints, (1) Include the set of flow conservation constraints, link-path incidence relationships and nonnegativity conditions. For a given upper-level decision variable y, we can obtain an equilibrium flow x by solving the lower-level problem, the upper-level decision-maker redesign the vector y. Repeat.

This process, at last the system may reach equilibrium. The vector of cost functions of links will in general have asymmetric Jacobian matrix, therefore, the lower-level problem is an asymmetric user equilibrium assignment problem which dose not have an equivalent optimization formulation. An equivalent variational inequality can be used to describe the user equilibrium assignment problem.

SOLUTION ALGORITHM BASED ON SENSITIVITY ANALYSIS

The algorithm described here is based on a descent-type algorithm. The algorithm relies on derivative information about the lower level problem with respect to the upper level decision variables in determining the search direction of the upper level problem. The calculation of a derivative is the most crucial part of a descent-type algorithm.

In order to solve the bi-level model proposed above, we need to consider the changes of Equilibrium link flows resulting from the changes of vector y except satisfying the constraints in the model. Sinc.,

$$\sum_{a\in A} x_a t_a(x,y) + \theta \sum_{a\in A} G_a(y_a)$$

Is in a nonlinear and implicit function form, the prediction of variations in link ?ows accompanied by changes of y cannot be carried out explicitly. The approach to avoid this difficulty is to formulate a linear approximation of:

$$\sum_{a\in A} x_a t_a(x,y) + \theta \sum_{a\in A} G_a(y_a)$$

At current point based on its derivative with respect to y. Therefore, we must compute the derivative and the calculation of this derivative is the crucial part for solving the model. The derivative information will be obtained by following the method of sensitivity analysis.

According to Ngnyen and Dupuis, the necessary conditio (f^*,λ,μ) ns of the lower problem are that there exists vector at y = 0 satisfying a system of following equations:

$$T(f^*,0) - \lambda - \Lambda^T \mu = 0 \tag{10}$$

$$\lambda^T f^* = 0 \tag{11}$$

$$\Lambda f^* - d = 0 \tag{12}$$

$$\lambda \ge 0, f^* \ge 0 \tag{13}$$

Where, T is the vector of cost function of path flows, f^* is the vector of equilibrium path flows, λ, μ is the vector of corresponding Lagrange multipliers, respectively.

The solutions of lower-level problem in terms of link flows are unique if the Jacobian matrix of cost function is positive but the solutions in terms of path flows are not unique. Because the system of equations is described in terms of path flows, this non-uniqueness of the solution in terms of path flows makes the derivative calculation impossible. Two methods could be useful. One is based on the restriction approach proposed by Tobin and Friesz, in which an equivalent restricted problem for the network equilibrium problem is developed which has the desired uniqueness properties required by the standard sensitivity analysis method for a nonlinear programming problem (Qui and Magnanti, 1989). The other one is proposed by Qui and Magnanti who developed a more straightforward approach that does not require uniqueness of the equilibrium solution; thus the restrictive assumptions on the selection of active paths could be avoided. It should be pointed out that although the approach of Qui and Magnanti is desirable, it is designed for the calculation of directional derivatives of the network equilibrium flow for any predetermined directions of the perturbation parameters. In addition, the derivative calculation would require solution of another variational

inequality that can be interpreted as a network equilibrium problem with path flows restricted between upper and lower bounds. To overcome the difficulty of the non-uniqueness, Tobin and Friesz (A.R. Conn, *et al.*) proposed a restriction approach, in which an equivalent restricted problem is developed which has the desired uniqueness properties.

The restriction approach is to select a nondegenerate extreme point in the region Ω. Let $f^*>0$ be a nondegenerate extreme point in Ω, it can be shown that under the assumption of strict positive link flows, the nonbinding nonnegative constraints in (13) remain nonbinding near y = 0 and can be eliminated without changing the solution near y = 0. The Lagrange multipliers associated with the nonbinding constraints equal zero and remain zero near y = 0. The corresponding derivatives of these multipliers with respect to y are equal to zero. When we consider only the nondegenerate extreme point of positive path flow solutions and delete the nonbinding constraints, the system of equations then reduces to:

$$\lambda \ge 0, f^* \ge 0 \tag{14}$$

$$\Lambda^0 f^{0*} - d = 0 \tag{15}$$

where, the superscript 0 represents the corresponding reduced vectors and matrixes.

Differentiating both sides of the system of Eq. 14 and 15 with respect to y, we obtain:

$$\begin{pmatrix} \frac{df}{dy} \\ \frac{d\mu}{dy} \end{pmatrix} = \begin{pmatrix} \nabla_f T^0(f^*,0) & -\Lambda^{0T} \\ \Lambda^0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_y T^0(f^*,0) \\ 0 \end{pmatrix}$$

It can be shown that the Jacobian M of systems in Eq. 14 and 15 with respect to f^0, μ is nonsingular.

Assume:

$$M^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

It is easily shown that:

$$B_{11} = N^{-1}\left[-E - \Lambda^{0T}(\Lambda^0 N^{-1}\Lambda^{0T})^{-1}\Lambda N^{-1}\right]$$

$$B_{12} = N^{-1}\Lambda^{0T}(\Lambda^0 N^{-1}\Lambda^{0T})^{-1}$$

$$B_{21} = -(\Lambda^0 N^{-1}\Lambda^{0T})^{-1}\Lambda N^{-1}$$

$$B_{22} = (\Lambda^0 N^{-1}\Lambda^{0T})^{-1}$$

And:

$$\nabla_f T^0(f^*, 0) = \Delta^{0T}\nabla_x t^0(x^*, 0)\Delta^0$$

$$\nabla_y T^0(f^*, 0) = \Delta^{0T}\nabla_y t^0(x^*, 0)$$

$$\frac{dx}{dy} = \Delta \cdot \frac{df}{dy}\frac{df}{dy} = -B_{11}\nabla_y T^0(f^*, 0)$$

where, Δ^0 is the reduced link/path incidence matrix.

Let us suppose that we had an initial solution y^* and the corresponding equilibrium link flow being f^* (It can be obtained by solving the lower-level problem). Let:

$$\phi(x) = \sum_{a\in A} x_a t_a(x, y) + \theta\sum_{a\in A} G_a(y_a)$$

Then:

$$\phi(x) \approx \phi(x^*) + (\nabla_x\phi(x)\cdot\frac{dx}{dy})^T(y - y^*) \quad (16)$$

When 16 is taken as objective function, we can solve the upper-level model by trust-region algorithm (A.R. Conn *et al.*). Then, according to the optimal variable y solved in upper-level problem, we can solve lower-level problem again and obtain new equilibrium link flows. We can obtain new optimal y values by repeating the basic idea above. After computing again and again, we will obtain the optimal solutions for bi-level model.

In this study, we applied the trust region method to solve the upper model, Trust region methods have proved to be very successful for unconstrained optimization problems, The trust region method is one of the most important methods for problem arising in some important mathematical problems such as nonlinear complementarity and variational inequalities problems. In trust region methods, the initial trust region radius plays an important role since it determines the direction and stepsize of the current iteration. However, in the traditional trust region methods, the initial radius is given randomly which effects the efficiency of the algorithm dramatically. So many self-adaptive trust region methods were proposed in recent years.The main idea of these self-adaptive trust region methods is that the initial trust region radius is controlled by the gradient of the current point. Numerical tests showed that the self-adaptive method is encouraging.

The solution algorithm can be stated as follows:

- **Step 1:** Initialization. Determine an initial value y^0; set k = 0
- **Step 2:** Solving the lower-level problem. Solve the lower-level problem with Predictor-corrector smoothing method for given y^k, obtain link/path incidence matrix (Δ), O-D/path incidence matrix (Λ) and optimal link flows y^{*k}
- **Step 3:** Derivative calculation. Calculate:

$$\frac{df}{dy^k} \text{ and } \frac{dx}{dy^k}$$

using the sensitivity analysis method

- **Step 4:** Solving the upper-level problem by trust-region algorithm. To solve the problem as follows:

$$\min \phi(x) + (\nabla_x\phi(x)\cdot\frac{dx}{dy})^T(y - y^k)$$

s.t.:

$$\|y - y^k\|_\infty \le \tau_k y \ge 0.$$

Above is a linear programming problem which can be easily solved.

- **Step 5:** Convergence check. if $\|y^k - y^{k+1}\| \le \varepsilon$ then stop; otherwise let k = k+1 and go to Step 2 ($\varepsilon$is a convergence tolerance)

NUMERICAL EXAMPLES

The test network is shown in Fig. 1, with associated data presented below. This problem has two O/D pairs (1, 2) and (2, 1), 6 nodes and 16 links. Table1 shows the solutions of P2 with methods proposed in this study:

$$t(x) = 10^{-3}Ax + bb = (2, 2, 1, 1.5, 3, 2, 1, 1, 1, 1, 2, 2.5, 3, 4, 1, 2)^T$$

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$b = (2, 2, 1, 1.5, 3, 2, 1, 1, 1, 1, 2, 2.5, 3, 4, 1, 2)^T$$

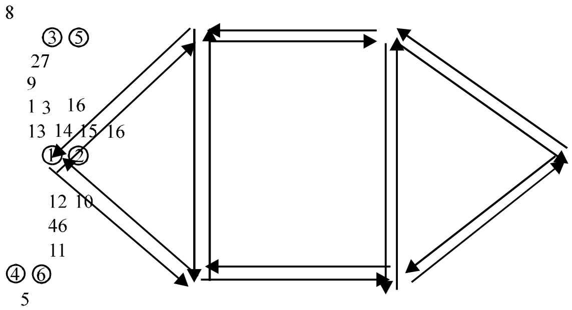

Fig. 1: Network for example

$$
A=\begin{pmatrix}
10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&15&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&18&0&0&0&2&0&0&0&0&0&0&0&0&1\\
0&0&0&10&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&20&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&16&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&10&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&12&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&20&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&15&0&0&0&0&0&0\\
0&0&0&0&3&0&0&0&0&0&10&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&13&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&10&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&20&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&18&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&20
\end{pmatrix}
$$

$$
\Delta=\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0\\
1&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&1&0&1&1&1&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1\\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&1&1&0&0&0&1&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&0&0&0&0&1&1
\end{pmatrix}
$$

In this study, we present a method for solving network design problem with equilibrium constrains the traffic equilibrium problem. The method starts from an initial point and successively generates points that converge to a solution. The method uses the gradients of the upper level objective function with respect to the capacity improvements to generate next points. The derivative of the link flow with respect to the capacity improvements is utilized in the calculation of the gradients of the upper level objective function. From Table.1, we can see that the capacity of some links are enhanced, the performance of the total network are improved. And the numerical results show that the method is promising.

Table. 1: Result of the algorithm

Demand	link	improvement
100	y_2	2.73
	y_3	0
	y_8	3.12
	y_9	0
	y_{15}	2.43
	y_{16}	1.97
	Z	5793.83
200	y_2	23.22
	y_3	8.31
	y_8	9.45
	y_9	5.75
	y_{15}	16.43
	y_{16}	13.17
	Z	6347.48

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