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Three-channel Symmetric Tight Frame Wavelet Design Method

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Abstract: A novel technique was presented for designing three-channel tight wavelet frame filters, motivated by the discovery of the property of the Selesnick tight frame wavelet. The filters had a linear phase response. The parametric Bernstein polynomials were utilized in the construction of the tight frame wavelets. The perfect reconstruction was inherent in the structure of the filters. The filters are nearly shift invariant and can be readily obtained by the spectral factorization method. Our method has lower computational complexity.

Key words: Bernstein polynomials, tight wavelet frame, filters, refinement equation, symmetric, antisymmetric

INTRODUCTION

Wavelet frames have drawn many attentions and become a powerful theoretical tool researching signal analysis (Ron and Shen, 1997; Chui and He, 2000; Selesnick, 2001a, b; Kingsbury, 2001; Petukhov, 2001; Chui *et al*., 2002; Daubechies *et al*., 2003; Jiang, 2003; Petukhov, 2003; Selesnick and Abdelnour, 2004; Abdelnour and Selesnick, 2005; Han and Mo, 2005; Selesnick *et al*., 2005; Han, 2009; Chaudhury and Unser, 2010). Specially, the tight wavelet frame with two generators exists. For example, Chui and He (2000) and Petukhov (2003) demonstrated a general characterization of tight wavelet frames. Based on "Unitary Extension Principle (UEP)" and "Oblique Extension Principle (OEP)", some scholars obtained tight wavelet frames with two generators (Chui and He, 2000; Selesnick, 2001a, b; Petukhov, 2001; Daubechies *et al*., 2003; Jiang, 2003; Petukhov, 2003; Han and Mo, 2005). Selesnick and Abdelnour (2004) used the UEP approach for the construction of wavelet tight frames with two (anti-) symmetric wavelets.

Caglar and Akansu (1993) and Rajagopal and Roy (1987) provided the design of Perfect Reconstruction Quadrature Mirror Filters (PRQMF) through the use of Bernstein polynomials. The parametric Bernstein polynomials were also utilized in the construction of the biorthogonal filter banks (Tay, 2005, 2008, 2010) and four-channel filter banks (Zhao and Zhao, 2012).

In this study, a parametric design technique is introduced based on Bernstein polynomial approximation for (anti-)symmetric tight wavelet frames with the two generators. Some definitions and notations are now recalled that will be used in this study. A scaling function $\phi(x)$ satisfies the refinement equation:

$$\phi(x) = \sqrt{2}\sum_{k\in Z} h_0(k)\phi(2x-k) \tag{1}$$

where, h_0 is a finitely supported sequence on Z, called the mask for the scaling function ϕ. The Fourier series of a sequence h_0 on Z is defined to be:

$$H_0(\omega) := \sum_{k\in Z} h_0(k)e^{-ik\omega}, \omega\in\mathbb{R} \tag{2}$$

The Fourier transform of a function $f\in L^1(\mathbb{R})$ is defined by:

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x)e^{-ix\omega}dx, \omega\in\mathbb{R} \tag{3}$$

And can be naturally extended to other function spaces such as $L^2(\mathbb{R})$. In terms of the Fourier transform, the refinement in Eq. 1 can be rewritten as:

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}H_0(\omega/2)\hat{\phi}(\omega/2), \omega\in\mathbb{R} \tag{4}$$

Wavelets ψ are defined through the dilation equations:

$$\psi^l(x) = \sqrt{2}\sum_{k\in Z} h_l(k)\phi(2x-k) \tag{5}$$

And two-scale symbols:

$$H_l(\omega) = \sqrt{2}\sum_{k\in Z} h_l(k)e^{-ik\omega}, l=1,2 \tag{6}$$

Using multirate system theory, the perfect reconstruction requirements of the three-channel tight wavelet frame filter bank reduces to:

$$H_0(z)H_0(z^{-1})+H_1(z)H_1(z^{-1})+H_2(z)H_2(z^{-1}) = 2 \tag{7}$$

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$$H_0(-z)\,H_0(z^{-1})+H_1(-z)\,H_1(z^{-1})+H_2(-z)\,H_2(z^{-1}) = 0 \quad (8)$$

where, $z = e^{i\omega}$. The polyphase components are defined so that:

$$H_i(z) = H_{i0}(z^2)+z^{-1}H_{i1}(z^2),\ \text{for}\ i = 0,1,2 \quad (9)$$

Let f(x) be defined on the interval [0,1]. The Nth ($N\geq 1$) order Bernstein polynomial approximation to f(x) is expressed as Davis (1963):

$$B_N(f,x) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x^k (1-x)^{N-k} \quad (10)$$

The Bernstein polynomials provide simultaneous approximations of a function and its derivatives and the values of f(x) at these (N+1) points only are used to evaluate the polynomial B_N (f,x). Now consider the approximation problem of filters $\{H_0(z), H_1(z), H_2(z)\}$ using the Bernstein polynomial.

BOTH ANTISYMMETRIC FILTER BANK

Throughout this study, we always assume that a scaling function ϕ satisfies $\hat{\phi}(0) = 1$ and sequences on Z are assumed to be real-valued. We are only interested in compactly supported real-valued scaling function ϕ in the spaces $L^2(\mathbb{R})$. The lowpass filter $h_0(n)$ symmetric:

$$h_0(n) = h_0(L-1-n) \quad (11)$$

With the even-length L. By (4), using the scaling function ϕ satisfies $\hat{\phi}(0) = 1$, the mask h_0 satisfies:

$$H_0(0) = \sum_{k\in Z} h_o(k) = \sqrt{2} \quad (12)$$

When L is even, according to (12), the polyphase components of $H_0(z)$ are satisfy:

$$H_{01}(z) = z^{-(L/2-1)}H_{00}(1/z) \quad (13)$$

Selesnick and Abdelnour (2004), did not find the algorithm to implement the filters for Case II. As the author's comments in Selesnick and Abdelnour (2004) "For Case II, we do not know how to obtain the polynomials A(z) and B(z) directly from the lowpass filter $H_0(z)$ using a simple root selection procedure as in Case I". In this paper, our designs begin with the construction of A(z) and B(z) using parametric Bernstein polynomial. In accordance with the property of Case II (Selesnick and Abdelnour, 2004), $\{h_0(n),h_1(n),h_2(n)\}$ satisfy:

$$h_1(n) = -h_1(L-1-n) \quad (14)$$

$$h_2(n) = -h_2(L-3-n) \quad (15)$$

The polyphase components satisfy:

$$H_{11}(z) = -z^{-(L/2-1)}H_{10}(1/z) \quad (16)$$

$$H_{21}(z) = -z^{-(L/2-2)}H_{20}(1/z) \quad (17)$$

Lemma (Selesnick and Abdelnour, 2004):. The filters $\{h_0,h_1,h_2\}$ with symmetries (11) (14) and (15) satisfy the perfect reconstruction conditions if their polyphase components are given by:

$$H_{00}(z) = \frac{z^{-1}}{\sqrt{2}}A^2(z) + \frac{1}{\sqrt{2}}B^2(z) \quad (18)$$

$$H_{10}(z) = \frac{z^{-1}}{\sqrt{2}}A^2(z) - \frac{1}{\sqrt{2}}B^2(z) \quad (19)$$

$$H_{20}(z) = \sqrt{2}A(z)B(z) \quad (20)$$

With (13) (16) and (17), where A(z), B(z) satisfy:

$$A(z)A(1/z)+B(z)B(1/z) = 1 \quad (21)$$

Let us consider now desired F(x) and G(x), $0\leq x\leq 1$, with Bernstein polynomials which satisfies:

$$F(x)+G(1-x) = 1,\ F(x)\geq 0,\ G(x)\geq 0 \quad (22)$$

And F(x) = B_{N-1} (F, x) as:

$$B_{N-1}(F,x) = \sum_{k=0}^{N-1} \beta_k \binom{N-1}{k} x^k (1-x)^{N-1-k} \quad (23)$$

And G (x) = B_{N-1} (G,x) as:

$$B_{N-1}(G,x) = \sum_{k=0}^{N-1} (1-\beta_k) \binom{N-1}{k} x^k (1-x)^{N-1-k} \quad (24)$$

where, parameters β_k, k = 0,1...,N-1 will be designed in accordance with practical needs. The polynomials can be transformed into z-transform filter function by the following substitution: $x = -1/4z(1-z^{-1})^2 = \sin^2(\omega/2)$. Then the B_{N-1} (F, x) can be expressed as:

$$R(z) = \frac{1}{4^{N-1}} z^{N-1} \cdot \{\sum_{k=0}^{N-1} \beta_k (-1)^k \binom{N-1}{k} \cdot (1+z^{-1})^{2(N-1-k)} (1-z^{-1})^{2k}\} \quad (25)$$

Similarly, the B_{N-1} (G, x) become:

$$R_1(z) = \frac{1}{4^{N-1}} z^{N-1} \cdot \{\sum_{k=0}^{N-1} (1-\beta_k) \cdot (-1)^k \binom{N-1}{k} \cdot (1+z^{-1})^{2(N-1-k)} (1-z^{-1})^{2k}\} \quad (26)$$

By (22), we can obtain $R(\omega)+R_1(\omega) = 1$. Let $R(\omega) = |A(\omega)|^2$ and $R_1(\omega) = |B(\omega)|^2$, so $R(\omega)$ and $R_1(\omega)$ satisfies the condition (21) that can be use to design $A(\omega)$ and $B(\omega)$, respectively. The $A(\omega)$ is obtained from the magnitude square function R(z) via spectral factorization method (Abdelnour and Selesnick, 2005). Similarly, $B(\omega)$ can be obtained from the magnitude square function $R_1(z)$. We will first obtain A(z) and B(z) by Bernstein Polynomials through (25) and (26). And then utilized (18)--(20), both anti-symmetric filters $h_1(n)$ and $h_2(n)$ were obtained by (13) (16) and (17).

The values β_k, k = 0,1...,N-1, determine the values of the filters. We can use the parameters β_k, k = 0,1...,N-1, in (25) and (26) designed to meet the requirements of the filters. To ensure the magnitude square functions $R(\omega)\geq 0$ and $R_1(z)\geq 0$, we desire the parameters in (25) and (26) to satisfy:

$$1 \geq \beta_k \geq 0, \text{ for, } k = 0,1...,N-1 \quad (27)$$

The $A^2(z)$ is the vector of length 2N-1 which convolves vectors A(z) by itself. The $z^{-1}/\sqrt{2}\, A^2(z)$ shift $A^2(z)$ by 1 samples. It follow from the condition (18) that length of the $H_{00}(z)$ is even 2N, then L/2 is even. In accordance to the conditions (9) (13) and (16)-(20), $H_0(0) = \sqrt{2}$, $H_0(\pi) = 0$, $H_1(0) = 0$, $H_2(0) = 0$.

In summary, the design procedure goes as follows:

- Choose the β_k, k = 0,1,...,N-1
- Find R(z) and R_1 (z) using (25) and (26)
- Knowing $R(z) = A(z)\, A(z^{-1})$ and $R_1(z) = B(z)\, B(z^{-1})$ Find A(z) and B(z) from R(z) and R_1 (z) using spectral factorization
- The filters $\{H_{00}(z), H_{10}(z), H_{20}(z)\}$ are obtained by (18)-(20)
- The filters $\{h_0,h_1,h_2\}$ are obtained by (13) (16) and (17)

Design examples

Example 1: To illustrate an example of parametric Bernstein polynomials design, taking on $\{\beta_0,\beta_1,\beta_2,\beta_3\}$ = {0,0.125,0.25,0.5}. The filterbank listed in Table 1. We obtain the filters depicted in Fig. 1. The resulting scaling function and wavelets are shown in Fig. 2. The antisymmetric wavelets obtained by these filters. Additionally, the filters are nearly shift-invariant, as can readily be seen in Fig. 3.

Example 2: With N = 5, taking on $\{\beta_0,\beta_1,\beta_2,\beta_3,\beta_4\}$ = {0,0.0001,0.0001,0.5,0.5} in (25) and (26). The filterbank listed in Table 2. The resulting calling function and wavelets are shown in Fig. 4. The antisymmetric wavelets obtained by these filter.

SYMMETRIC AND ANTISYMMETRIC WAVELETS AND APPLICATIONS

The algorithm can be similarly given to implement the filters for case I in Selesnick and Abdelnour (2004).

Example 3: Using Lemma 1 in Selesnick and Abdelnour (2004), the filters for Case I in Selesnick and Abdelnour (2004) can be gotten. With N = 4 in (25) and (26), taking on $\{\beta_0,\beta_1,\beta_2,\beta_3,\beta_4\}$ = {0.5,0.5,0.5,0.1}. These results in the filter bank listed in Table 3. We obtain the filters depicted

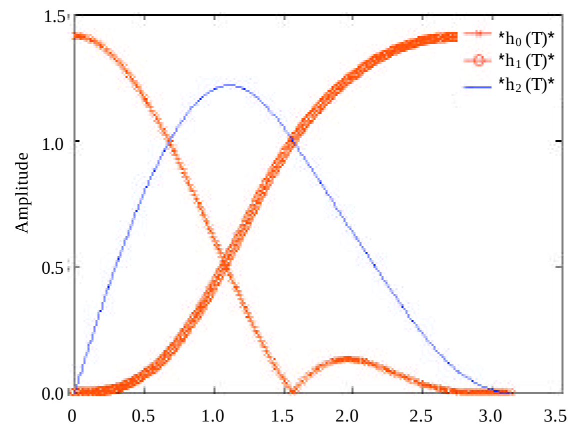

Fig. 1: Example 1: Amplitude spectrum

Table 1: Example 1: filterbank $\{h_0(n), h_1(n), h_2(n)\}$ coefficients

$h_0(n)$	$h_1(n)$	$h_2(n)$
0.00000358483272	0.00000358483272	-0.00001920473488
0	0	0
-0.00001848881978	-0.00006993080749	0.00021303998976
0	0	0
0.00034200889851	0.00084890447015	-0.00291993157245
0	0	0
0.00351938735995	-0.00365759916324	0.00651554007087
0	0	0
-0.01969572721106	0.01503640218043	-0.04077379797135
0.07415532399177	0.07415532399177	-0.39726632931866
0.27589716806134	0.07788285396250	-0.36028196620721
0.37290352407309	-0.69121788970325	0.36028196620721
0.37290352407309	0.69121788970325	0.39726632931866
0.27589716806134	-0.07788285396250	0.04077379797135
0.07415532399177	-0.07415532399177	0
-0.01969572721106	-0.01503640218043	-0.00651554007087
0	0	0
0.00351938735995	0.00365759916324	0.00291993157245
0	0	0
0.00034200889851	-0.00084890447015	-0.00021303998976
0	0	0
-0.00001848881978	0.00006993080749	0.00001920473488
0	0	0
0.00000358483272	-0.00000358483272	0

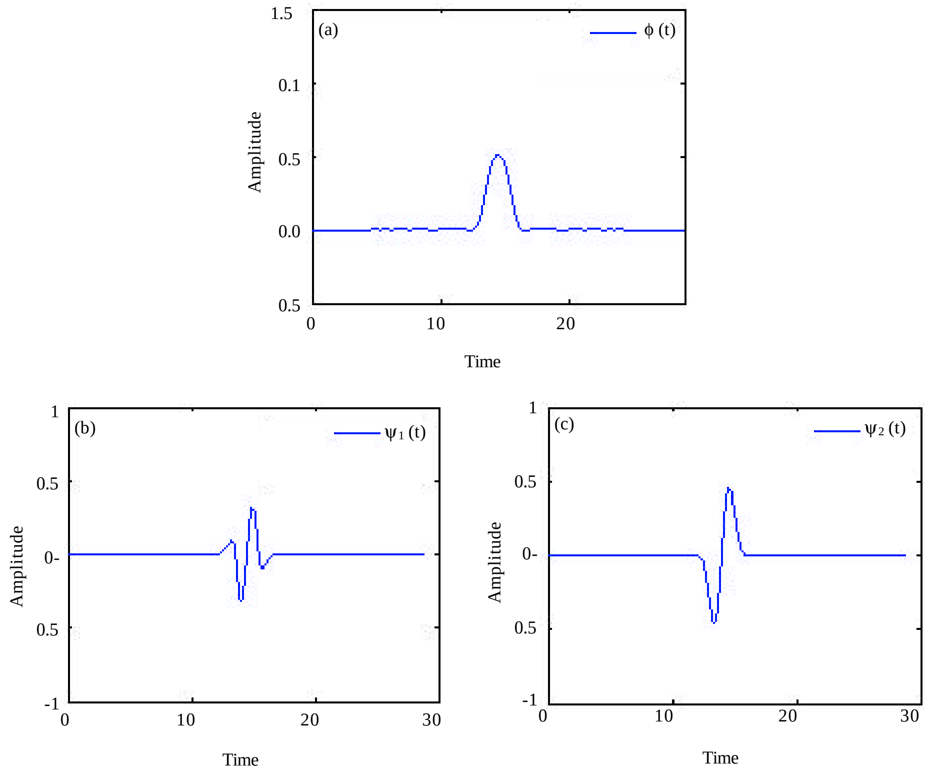

Fig. 2: Example 1: Wavelets and scaling function

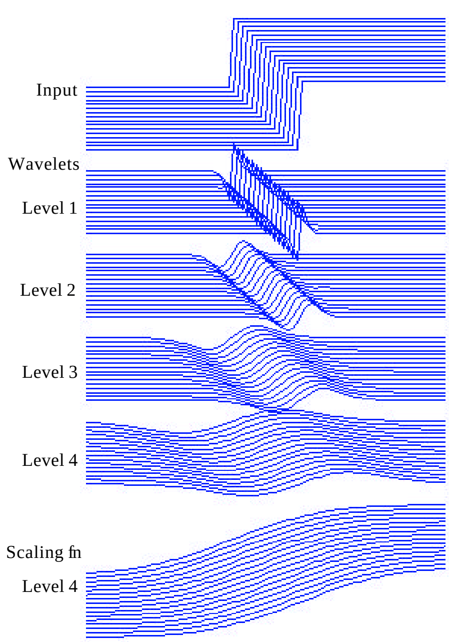

Fig. 3: Example 1: Wavelet and scaling function components at level 1-4 of 16 shifted step responses

Table 2: Example 2: filterbank $\{h_0(n), h_1(n), h_2(n)\}$ coefficients

$h_0(n)$	$h_1(n)$	$h_2(n)$
0.00003207247266	0.00003207247266	-0.00035575077458
0	0	0
0.00080328014939	-0.00116972977356	0.00113451740882
0	0	0
-0.00061450277297	0.00069818287603	0.00311045153226
0	0	0
-0.01004823930236	0.01446025722166	-0.01901086042584
0	0	0
0.02759935749287	-0.01929039585377	-0.03922883694418
0	0	0
-0.00505475145922	-0.05309466491159	0.07780719956436
0	0	0
-0.12736894291349	0.10939850343840	0.19706591893547
0.01739632443211	0.01739632443211	-0.19296160784427
0.35045677677333	0.05720046189564	-0.41348424714058
0.45390540631424	-0.61626841825320	0.41348424714058
0.45390540631424	0.61626841825320	0.19296160784427
0.35045677677333	-0.05720046189564	-0.19706591893547
0.01739632443211	-0.01739632443211	0
-0.12736894291349	-0.10939850343840	-0.07780719956436
0	0	0
-0.00505475145922	0.05309466491159	0.03922883694418
0	0	0
0.02759935749287	0.01929039585377	0.01901086042584
0	0	0
-0.01004823930236	-0.01446025722166	-0.00311045153226
0	0	0
-0.00061450277297	-0.00069818287603	-0.00113451740882
0	0	0
0.00080328014939	0.00116972977356	0.00035575077458
0	0	0
0.00003207247266	-0.00003207247266	0

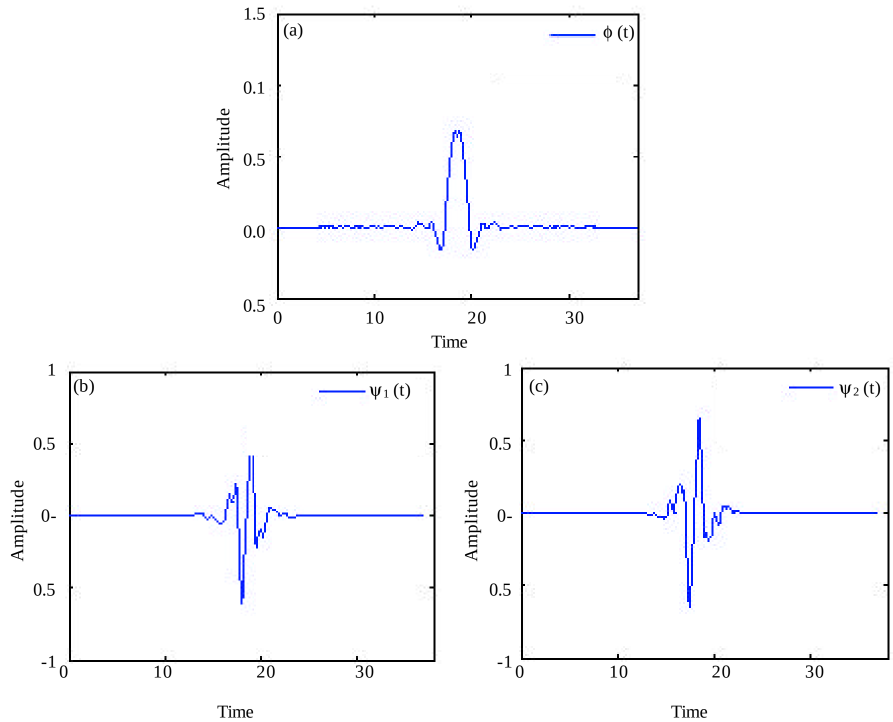

Fig. 4: Example 2: Wavelets and scaling function

Table 3: Example 3: filterbank {$h_0(n)$, $h_1(n)$, $h_2(n)$} coefficients

$h_0(n)$	$h_1(n)$	$h_2(n)$
-0.00656125013401	0.33614524813620	-0.00006403482358
0.01190702966726	-0.61001963959815	0.00011620720571
0.03633026771432	0.21412397016314	0.00074992392711
-0.07695399720988	0.17618032856283	-0.00146851026747
-0.07946949036865	-0.04488208552041	-0.00400069342912
0.21554506707272	-0.08591563214915	0.00862063017850
0.60630915444480	-0.01265545989567	0.02306974751006
0.60630915444480	0.02306974751006	-0.01265545989567
0.21554506707272	0.00862063017850	-0.08591563214915
-0.07946949036865	-0.00400069342912	-0.04488208552041
-0.07695399720988	-0.00146851026747	0.17618032856283
0.03633026771432	0.00074992392711	0.21412397016314
0.01190702966726	0.00011620720571	-0.61001963959815
-0.00656125013401	-0.00006403482358	0.33614524813620

Table 4: Example 3: filterbank {$h_0(n)$, $h_1(n)$, $h_2(n)$} coefficients

$h_0(n)$	$h_1(n)$	$h_2(n)$
-0.00656125013401	-0.00004527945798	0.00004527945798
0.01190702966726	0.00008217090318	-0.00008217090318
0.03633026771432	0.23822086071498	0.23716030812651
-0.07695399720988	-0.43238741738520	-0.43031063024846
-0.07946949036865	0.14857959386376	0.15423742877013
0.21554506707272	0.13067401109577	0.11848259898114
0.60630915444480	-0.01542365212065	-0.04804920192990
0.60630915444480	-0.06970028761386	-0.05180276459133
0.21554506707272	-0.06970028761386	0.05180276459133
-0.07946949036865	-0.01542365212065	0.04804920192990
-0.07695399720988	0.13067401109577	-0.11848259898114
0.03633026771432	0.14857959386376	-0.15423742877013
0.01190702966726	-0.43238741738520	0.43031063024846
-0.00656125013401	0.23822086071498	-0.23716030812651
0	0.00008217090318	0.00008217090318
0	-0.00004527945798	-0.00004527945798

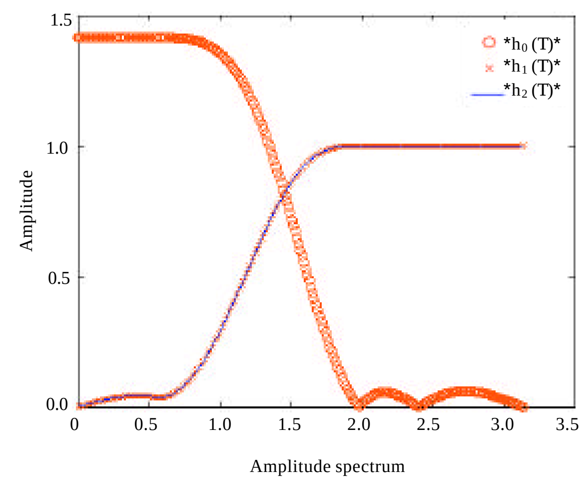

Fig. 5: Example 3: Amplitude spectrum

in Fig. 5. The resulting scaling function and wavelets are shown in Fig. 6. Additionally, the filters are nearly shift-invariant, as can readily be seen in Fig. 7

We can also use Eq. 11 and 12 (Selesnick and Abdelnour, 2004) to obtain a wavelet tight frame with (anti-) symmetric wavelets where d = 1. The filters are listed in Table 4. The symmetric and antisymmetric wavelets obtained by these filters are illustrated in Fig. 8. Additionally, the filters are nearly shift-invariant, as can readily be seen in Fig. 9.

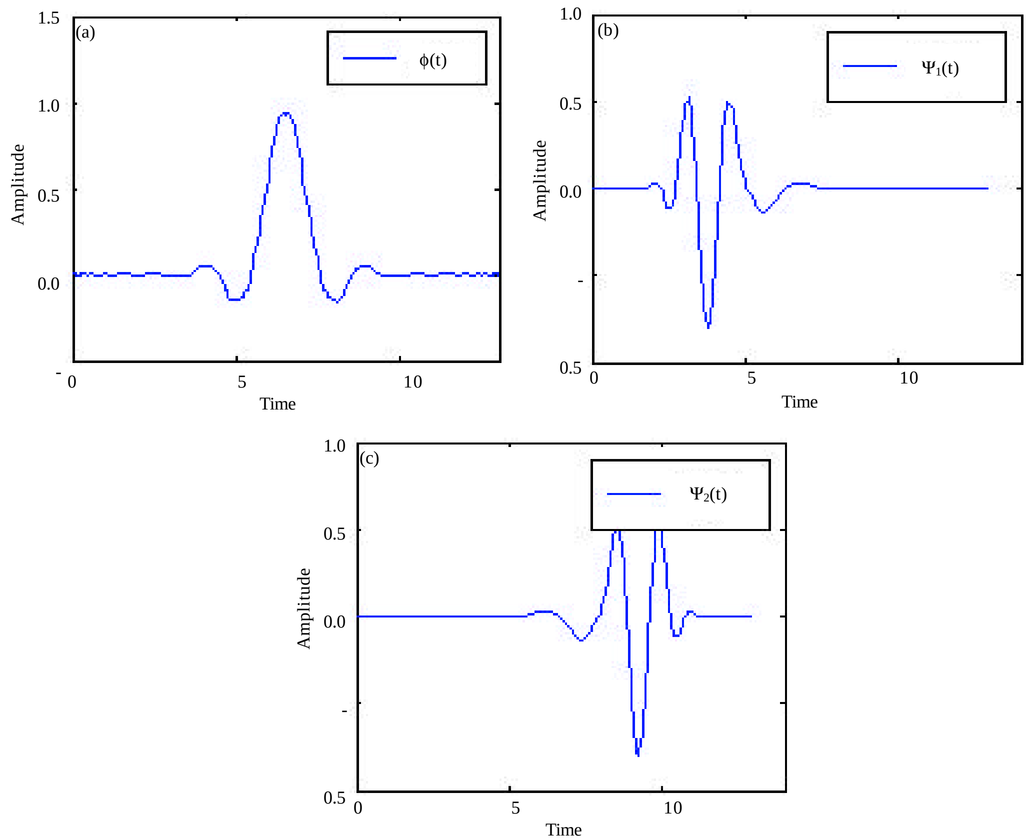

Fig. 6: Example 3: Wavelets and scaling function

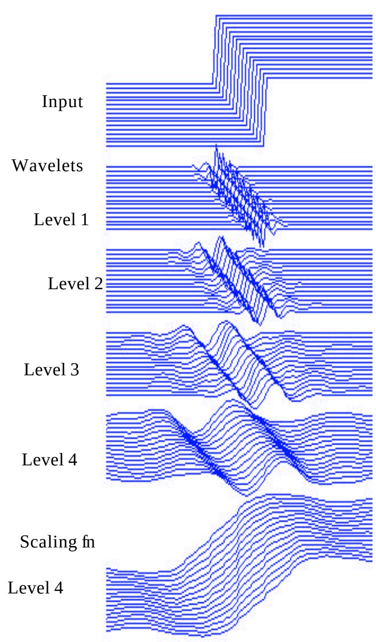

Fig. 7: Example 3: Wavelet and scaling function components at level 1-4 of 16 shifted step responses

To verify performance of our design filters, we consider in this section an example of applications of three-channel tight frame wavelets to the area of noise removal. We will consider a 512×512 Lean image and Barbara image subjected to additive white Gaussian noise. Denoising is using the soft threshold which is defined as:

$$\hat{x} = \text{sgn}(x)(|x| - T)_+ \tag{28}$$

Let s,d denote the original and the denoised image. The root mean-square (RMS) error is given by:

$$\text{RMS} = \sqrt{\frac{1}{N}\sum_k (s_k - d_k)^2}$$

where, the number of pixels is N. The PSNR in decibels is given by PSNR = 20 $\log_{10}$ (256/RMS). The number of all filterbanks decomposition level is 5 with symmetric extensions. Table 5 compared noise removal of the filterbanks with Table 3 to using Daubechies' length-8 wavelet and the example in Selesnick and Abdelnour (2004) from averaging over 10 realizations and the threshold $T = \sqrt{2}\sigma_n$ (Abdelnour and Selesnick, 2005), where, σ_n^2 is the noise variance estimate which is found

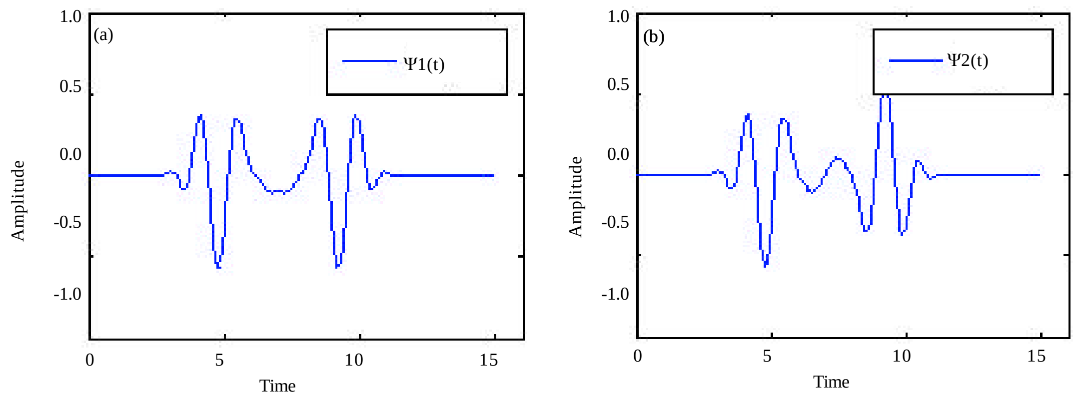

Fig. 8: Example 3: Symmetric and antisymmetric wavelets

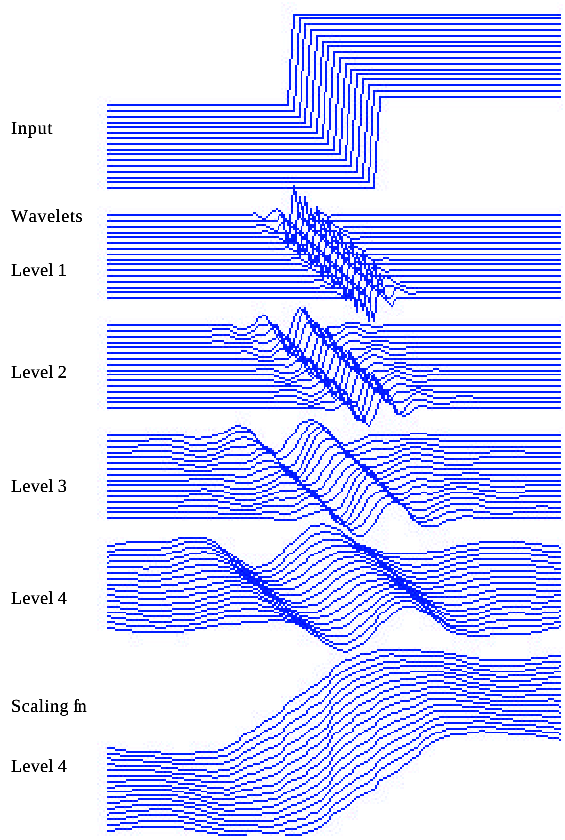

Fig. 9: Example 3 Wavelet and scaling function components at level 1-4 of 16 shifted step responses

from the high pass outputs. The denoised images obtained using the three-channel filterbanks in Selesnick and Abdelnour (2004) and using the proposed filterbanks are illustrated in Fig. 10(a-b) with the Gaussian noise standard deviation 25, respectively. Further work includes noise removal with the three-channel filterbanks, the example do only to verify performance of our design filters.

Fig. 10(a-b): (a) Denoised image using the filterbanks in Selesnick and Abdelnour (2004) PSNR = 28.2059 dB and (b) Denoised image using the filterbanks in Table 3 PSNR = 28.4709 dB

Table 5: PSNR values of denoised image

Lean	σ = 10	σ = 5	σ = 20	σ = 25	σ = 30
Noisy	28.18	24.65	22.14	20.17	18.62
Using Daubechies' length-8 wavelet	32.6652	30.6171	29.0888	27.8695	26.8329
Using the filterbanks in Selesnick and Abdelnour (2004)	32.2430	30.5219	29.2429	28.2059	27.3197
Using the filterbanks in Table 3	32.6127	30.8822	29.5479	28.4709	27.5883
Barbara	σ = 10	σ = 15	σ = 20	σ = 25	σ = 30
Noisy	28.16	24.63	22.14	20.18	18.62
Using Daubechies' length-8 wavelet	29.9910	27.9554	26.5387	25.4298	24.5472
Using the filterbanks in Selesnick and Abdelnour (2004)	28.2128	26.7356	25.7289	24.9441	24.3398
Using the filterbanks in Table 3	28.9529	27.3735	26.2484	25.4087	24.7267

CONCLUSION

A new approach for the design of a class of three-channel tight frame filter banks has been presented by using spectral factorization. The parametric Bernstein polynomials were employed and these ensured the perfect-reconstruction. Several design examples are presented to illustrate the effectiveness and flexibility of the design technique. Different from Selesnick and Abdelnour (2004) begin with the construction of a lowpass filter $h_0(n)$, our design begin with the construction of $A(\omega)$ and $B(\omega)$ and avoids selection of root and lowpass filter. The filters are nearly shift invariant and are constructed by appropriate parameters. Our method has lower computational complexity.

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