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Characterizations of Optimal Solution Set in Programming Problem under Inclusion Constrains

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Abstract: The Characterizations of the solution set in extremely problem under inclusion constrains:(P):

$$\min_{s.t. x \in C, 0 \in F(x).} f(x)$$

is considered in this study. When f is continuously convex and F is a set-valued map with convex graph, the Lagrange function of (P) is proved to be a constant on the solution set and this property is then used to derive various simple Lagrange multiplier-based characterizations of the solution set of (P).

Key words: Inclusion constrains, support functions, sub gradient, Solution set

INTRODUCTION

The characterization of optimal solution of a mathematical programming is an important study in optimization problems and it is fundamental for the development of solution methods.

Jeyakumar (Jeyakumar *et al.*, 2004) presented characterization of the solution sets of the following cone-constrained convex programming:
(P′)

$$\min_{s.t. x \in C, -g(x) \in K,} f(x)$$

where, X and Y are Bnanach Spaces, *C* is a closed convex subset of X, K is a closed convex cone in Y, f: X→R is a continuous convex function and g: X→Y is a continuous Knapping. The Lagrange multiplier, which is to identifying optimal solution for constrained optimization, is used to characterize the solution set of (P′). First; the author established that the Lagrange function of (P′) is constant on the solution set of (P′). Then, he used this elementary property to present various simple Lagrange multiplier-based characterizations of the solutions set of (P′). (Clarke, 1998). In this study, we consider the programming problem under inclusion constrains:
(P)

$$\min_{s.t. x \in C, 0 \in F(x).} f(x)$$

Suppose that C is closed convex subset of X,f is a continuous convex function and g is a set-valued mapping with convex graph. Obviously, constrain –g (x)∈K can be written 0∈g (x)+K. It is also easy to derive that g (x) +K is a map with convex graph on C, that is to say, the problem (P′) is a special case of problem (P) where F (x) = g (x)+K (Remark 2.1). We prove that the Lagrange function of problem (P) is constant on is solution set (Theorem 3.2). And we derive various characterizations of the solution set using the Lagrange multiplier (Theorem 3.3, Proposition 3.1, Proposition 3.2 and Proposition 3.3).

PRELIMINARES

Let X and Y to be banach spaces and X* and Y* are their dual spaces. Let C to be a nonempty closed subset of X. Suppose that f: X→R is a real-valued function and that g: X→Y is a set-valued mapping.

Definition 2.1: A function f is said to be satisfy a Lipschitz condition of rank K on a given set C provided that F is finite on C and satisfies:

$$|f(x) - f(y)| \le L\|x - y\|, \forall x, y \in C.$$

A function f is said to be Lipschitz near if it satisfies the Lipschitz condition on a neighborhood of x. A function f is said to be Locally Lipschitz on C if f is Lipschitz mere x for every x∈C.

Definition 2.2: Let f be Lipschitz of rank K near a given point x∈C. The generalized directional derivative of f at x in the direction v, denoted f^0 (x; v), is defined as follows:

$$f^{\circ}(x;v)=\limsup_{y\to x,t\downarrow 0}\frac{f(y+tv)-f(y)}{t}$$

where, of course y is a vector in X and t is a positive scalar.

Definition 2.3: The generalized gradient of f at x, denoted $\partial f(x)$ is defined to be the subset of X*:

$$\partial f(x)=\{x^*\in X^*: f^{\circ}(x;v)\geq \prec x^*,v\succ,\forall v\in X\}$$

Proposition 2.1: Let f be convex on *C* and Lipschitz near x∈C. Then the directional derivatives f′ (x; v) exist and we have f′ (x; v) = f° (x; v).A vector $x^*\in\partial f(x)$ iff:

$$f(y)-f(x)\geq\prec x^*,y-x\succ,\forall y\in C.$$

Definition 2.4: The tangent cone to C at x, denoted $T^C(x)$, is the set of all those v∈X satisfying:

$$d_C^{\circ}(x;v)=0$$

where, $d_C^{\circ}(x, v)$ is the distant function of C, given by:

$$d_C(x)=\inf\{\|x-c\|:c\in C\}$$

Definition 2.5: The normal cone to C at x, denoted $N_C(x)$, is defined the polarity of its tangent cone:

$$N_C(X)=(T_C(X))^{\circ}=\{x^*\in X^*:\prec x^*,v\succ\leq 0,\forall v\in T_C(x)\}$$

Proposition 2.2: Let C be convex. Then $T_C(x)$ = cl {λ(c-x): λ≥0, c∈C}

And:

$$N_C(x)=\{x^*\in X^*:\prec x^*,y-x\succ\leq 0,\forall y\in C\}$$

Definition 2.6: The graph of a set-valued map F to C is said to be convex if for any x_1, x_2 ∈C and μ∈ (0, 1), we have:

$$F(\mu x_1+(1-\mu)x_2)\supset\mu F(x_1)+(1-\mu)F(x_2)$$

Now consider the programming problem under inclusion constrains:

(P)
$$\min_{s.t.x\in C,0\in F(x),} f(x)$$

where, C is a closed convex subset of X: f: X→Y is a continuous convex function and g: X→Y is a set-valued mapping with convex graph.

Remark 2.1: Denote F (x) = g (x)+K

Then, the problem (P) degenerate into the cone-constrained convex programming (P′):

$$\min_{s.t.x\in C,-g(x)\in K,} f(x)$$

where, C is a closed convex subset of X, K is a closed convex cone in Y: f: X→R is a continuous convex function and g: X→Y is a continuous K mapping, that is, for any x_1, x_2∈C and μ∈(0, 1), we have:

$$\mu(g(x_1)+(1-\mu)g(x_2)-g(\mu x_1+(1-\mu)x_2)\in K$$

In fact, -g (x)∈K is 0∈g (x)+K. Denote F (x) = g (x)+K.

The only need to purify is that F (x) is a map with convex graph. For any x_1, x_2∈C, μ∈(0, 1) we have:

$$\begin{aligned}&\mu F(x_1)+(1-\mu)F(x_2)\\&=\mu(g(x_1)+K)+(1-\mu)(g(x_2)+K)\\&=\mu(g(x_1)+(1-\mu)g(x_2)+K\\&\subset g(\mu x_1+(1-\mu)x_2)+K+K\\&=-g(\mu x_1+(1-\mu)x_2)+K\\&=F(\mu x_1+(1-\mu)x_2)\end{aligned}$$

Then, the conclusion follows.

Since, F is a map with convex graph, its image F (X) is a convex set (Ekland, 2009). Suppose that the barrier cone of F:

$$Y_F^*=\{y^*\in Y^*:\inf_{y\in F(x)}\prec y^*,y\succ\succ-\infty\}$$

is closed and does not depend on x. This is the case, for example, when F is locally Lipschitz (Amahroq and Gadhi, 2003). For every $y^*\in Y_F^*$, the support functions of F is defined as follows:

$$C_F(y^*,\cdot)=\inf_{y\in F(\cdot)}\prec y^*,y\succ$$

For problem (P), let E and S are, respectively the feasible set and the solution set, that is:

$$\begin{aligned}&E=\{x\in C:0\in F(x)\},\\&S=\{x\in E:f(x)\leq f(y),\forall y\in E\}\end{aligned}$$

Remark 2.2: For problem (P′), that is, when F (x) = g (x)+K, we have:

$$C_F(y^*,\cdot)=\prec y^*,g(\cdot)\succ$$

In fact:

$$C_{g(x)+K}(y^*,x)=\inf_{y\in g(x)+K}\prec y^*,y\succ$$
$$=\inf_{y\in K}\prec y^*,g(x)+k\succ$$
$$=\prec y^*,g(x)\succ+\inf_{k\in K}\prec y^*,k\succ$$

If:

$$\inf_{k\in K}\prec y^*,k\succ 0$$

Then:

$$\inf_{k\in K}\prec y^*,k\succ-\infty$$

Which is conflict with:

$$y^*\in Y^*_{g(x)+K}=\{y^*\in Y^*:\inf_{y\in g(x)+K}\prec y^*,y\succ\succ-\infty\}$$

Sine $0\in K$, we have:

$$\inf_{k\in K}\prec y^*,k=0$$

Then, $C_{g(x)+K}(y^*, x) = \prec y^*, g(x)\succ$.

Denote:

$$L'(y^*,\cdot)=f(\cdot)+\prec y^*,g(x)\succ$$

And let E and S be respectively the feasible set and the solution set of problem (P'), that is:

$$E'=\{x\in C:0\in g(x)+K\},$$
$$S'=\{x\in E':f(x)\le f(y),\forall y\in E\}.$$

Proposition 2.3: If F is a map with convex graph and then $C_F(y^*,.)$ is a convex function on C.

Prove: Since is a map with convex graph, we have:

$$F(\mu x_1+(1-\mu)x_2)\supset\mu F(x_1)+(1-\mu)F(x_2),\forall x_1,x_2\in C,\mu\in(0,1)$$

Then:

$$\inf_{y\in F(\mu x_1+(1-\mu)x_2)}\prec y^*,y\succ \le \inf_{y\in(\mu F(x_1)+(1-\mu)F(x_2))}\prec y^*,y\succ$$
$$=\inf_{y\in\mu F(x_1)}\prec y^*,y\succ+\inf_{y\in(1-\mu)F(x_2)}\prec y^*,y\succ$$
$$=\mu\inf_{y\in F(x_1)}\prec y^*,y\succ+(1-\mu)\inf_{y\in F(x_2)}\prec y^*,y\succ$$

That is:

$$C_F(y^*,\mu x_1+(1-\mu)x_2)\le\mu C_F(y^*,x_1)+(1-\mu)C_F(y^*,x_2),\forall x_1,x_2\in C,\mu\in(0,1)$$

So, $C_F(y^*,.)$ is a convex function on C.

Definition 2.7: Problem (P) is said to be regular at a feasible point $\bar{x}$ if the system:

$$\begin{cases}C_F(y^*,\bar{x})=0\\ \partial C_F(y^*,\cdot)(\bar{x})=0\end{cases}\quad(2.1)$$

Admits a unique solution y*=0.

Let:

$$M(\bar{x})=\{\bar{\lambda}\in(Y_F^*)^+:0\in\partial L(\bar{\lambda},\bar{x})+N_C(\bar{x})+C_F(\bar{\lambda},\bar{x})=0\}$$

clearly, the Lagrange multiplier set M $(\bar{x})$ is nonempty if the problem (P) is regular at $\bar{x}$. Denote the Lagrange multiplier set of (P') (Jeyakumar *et al*., 2004) to be:

$$M'(\bar{x})=\{\bar{\lambda}\in K^+:0\in\partial L(\bar{\lambda},\bar{x})+N_C(\bar{x})\prec\bar{\lambda},g(\bar{x})\succ=0\}$$

CHARACTERIZATION OF SOLUTION SET

For problem (P), we have the following first order optimality condition.

Theorem 3.1: Suppose that the problem (P) is regular at $\bar{x}$. Then, $\bar{x}\in C$ if there exists a $\bar{\lambda}\in(Y_F^*)$, such that:

$$\begin{cases}0\in\partial L(\bar{\lambda},\bar{x})+N_C(\bar{x}),\\ C_F(\bar{\lambda},\bar{x})=0.\end{cases}\quad(3.1)$$

Prove: The necessary condition is proved in (Dien, 1985). We only prove the sufficiency.

It follows from (3.1) that there exists $\xi\in\partial L(\bar{\lambda},\bar{x})$ such that:

$$-\xi\in N_C(\bar{x})$$

From the definition of the convex sub differential and normal cone, we have:

$$L(\bar{\lambda},x)-L(\bar{\lambda},\bar{x})\ge\prec\xi,x-\bar{x},\forall x\in X$$

And:

$$\prec-\xi,x-\bar{x}\succ\le 0,\forall x\in C$$

Then:

$$L(\bar{\lambda},x)-L(\bar{\lambda},x)\ge 0,\forall x\in C$$

That is:

$$f(x)+C_F(\bar{\lambda},x)\ge f(\bar{x})+C_F(\bar{\lambda},x),\forall x\in C\quad(3.2)$$

Since:

$$0 \in F(x), \forall x \in E$$

It follows that:

$$C_F(\bar{\lambda}, x) = \inf_{y \in F(x)} \prec \bar{\lambda}, y \succ \leq 0, \forall x \in E \tag{3.3}$$

$$C_F(\bar{\lambda}, x) = 0 \tag{3.4}$$

And take (3.3), (3.4) to (3.2), we obtain:

$$f(x) \geq f(\bar{x}) - C_F(\bar{\lambda}, x) \geq f(\bar{x}), \forall x \in E \tag{3.5}$$

And so $\bar{x} \in S$.

Corollary 3.1 suppose that the problem (P) is regular at $\bar{x}$. Then, $\bar{x} \in S$ if there exists a $\bar{\lambda} \in (Y_F^*)$, such that:

$$\begin{cases} 0 \in \partial L'(\bar{\lambda}, \bar{x}) + N_C(\bar{x}), \\ \prec \bar{\lambda}, g(\bar{x}) \succ = 0. \end{cases}$$

Prove: By Remark 2.2, we have:

$$C_F(\bar{\lambda}, x) = \prec \bar{\lambda}, g(\bar{x}) \succ,$$
$$L'(\bar{\lambda}, \bar{x}) = f(\bar{x}) + \prec \bar{\lambda}, g(\bar{x}) \succ$$

Take it to Theorem 3.1 we can obtain the conclusion.

Remark 3.1: The proviso of Corollary 3.2, 3.3, 3.4, 3.5, 3.6 is similar to Corollary 3.1, so we omit them in this study.

From the first optimality condition, we can obtain the conclusion that the Lagrange function is constant on the solution set S.

Theorem 3.2: For problem (P), let $\bar{x} \in S$, $\bar{\lambda} \in M(\bar{x})$. Then:

$$C_F(\bar{\lambda}, x) = 0, \forall x \in S$$

That is, the Lagrange function L ($\bar{\lambda}$, x) is constant on S.

Prove: For any $\bar{x} \in S \subset E$, we have (3.5):

$$f(x) \geq f(\bar{x}) - C_F(\bar{\lambda}, x)$$

If C_F ($\bar{\lambda}$, s)$\prec$0

Then: f (x)$\succ$f ($\bar{x}$),

Which is conflict with f (x)$\succ$f ($\bar{x}$) so C_F ($\bar{\lambda}$, x) = 0 and $\bar{\lambda} \in M'$ ($\bar{x}$)

By the arbitrariness of $\bar{x}$, the Lagrange function L ($\bar{\lambda}$, x) is constant on S.

Corollary 3.2: For problem (P'), let $\bar{x} \in S'$, $\bar{\lambda} \in M'$ ($\bar{x}$). Then $\prec \bar{\lambda}$, g (x)$\succ$=0, $\forall x \in S'$.

That is, the Lagrange function L' ($\bar{\lambda}$, x) is constant on S'.

We derive now characterizations of solution set using the Lagrange multipliers as an application of the preceding theorem.

Theorem 3.3: For problem (P), let $\bar{x} \in S$, $\bar{\lambda} \in M$ ($\bar{x}$). Then:

$$S = \{x \in E : C_F(\bar{\lambda}, x) = 0, \exists \xi \in \partial f(x) \cap \partial f(\bar{x}), \xi(x - \bar{x}) = 0\}$$

Prove: Let:

$$S_1 = \{x \in E : C_F(\bar{\lambda}, x) = 0, \exists \xi \in \partial f(x) \cap \partial f(\bar{x}), \xi(x - \bar{x}) = 0\}$$

If $x \in S$, then f (x) = f ($\bar{x}$).

By Theorem 3.2, we have C_F ($\bar{\lambda}$, x) = 0.

Since $\bar{\lambda} \in M$ ($\bar{x}$), there exists $\xi \in \partial$ f ($\bar{x}$), $\zeta \in \partial C_F$ ($\bar{\lambda}$, $\bar{x}$) such that:

$$(\xi+\zeta)\ (x-\bar{x}) \geq 0 \tag{3.6}$$

Now, $\zeta \in \partial C_F$ ($\bar{\lambda}$, $\bar{x}$), gives us that:

$$0 = C_F(\bar{\lambda}, x) - C_F(\bar{\lambda}, \bar{x}) \geq \zeta(x - \bar{x})$$

Together with (3.6), this yields:

$$\xi\ (x-\bar{x}) \geq 0$$

On the other hand, since $\xi \in \partial f$ ($\bar{x}$), we have:

$$\xi(x - \bar{x}) \leq f(x) - f(\bar{x}) = 0$$

So, ξ (x-$\bar{x}$) = 0

We now prove that $\xi \in \partial f$ (x). For any $y \partial X$, we have:

$$\begin{aligned} & f(y) - f(x) \\ & = f(y) - f(\bar{x}) \\ & \geq \xi(y - \bar{x}) \\ & = \xi(y - x) + \xi(x - \bar{x}) \\ & = \xi(y - x). \end{aligned}$$

Thus, $\xi \in \partial f$ (x) Consequently, $x \in S_1$.

Conversely, if $x \in S_1$, the $x \in E$ and there exist $\xi \in \partial f$ (x), then:

$$f(\bar{x}) - f(x) \geq \xi(\bar{x} - x) = 0.$$

Since, $x \in S$, f (x) = f (x) = and so $x \in S$.

Corollary 3.3: For problem (P'), let $\bar{x}\in S'$, $\bar{\lambda}\in M'(\bar{x})$. Then:

$$S' = \{x \in E' : \prec \bar{\lambda}, g(x) \succ = 0, \exists \xi \in \partial f(x) \cap \partial f(\bar{x}), \xi(x - \bar{x}) = 0\}$$

Since (3.1) of Theorem 3.1 can change into:

$$\begin{cases} \partial L(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x})) \neq \varphi, \\ C_F(\bar{\lambda},\bar{x}) = 0. \end{cases}$$

We have:

Proposition 3.1: For problem (P), let $\bar{x}\in S$, $\bar{\lambda}\in M(\bar{x})$. Then:

$$\partial L(\bar{\lambda},x) \cap (-N_C(x)) = \partial L(\bar{\lambda},y) \cap (-N_C(y)), \forall x, y \in S.$$

That is, the function $\partial L(\bar{\lambda},\cdot) \cap (-N_C(.))$ is constant on S.

Prove: Let $\hat{x}$, $\tilde{x}\in S$. Suppose that $\xi\in X^*$ such that:

$$\xi \in \partial L(\tilde{x},\bar{\lambda}), \quad -\xi \in N_C(\tilde{x}).$$

By Theorem 3.2, we have:

$$C_F(\bar{\lambda},\hat{x}) = C_F(\bar{\lambda},\tilde{x})$$

And $\xi(\hat{x}, \tilde{x}) = 0$.

Moreover:

$$\xi(x - \hat{x}) = \xi(x - \tilde{x}) + \xi(\tilde{x} - \hat{x}) = \xi(x - \tilde{x}) \geq 0, \forall x \in C$$

Thus, $\xi\in -N_C(\hat{x})$.

And for all $x\in X$:

$$\begin{aligned} & f(x) + C_F(\bar{\lambda},x) - f(\hat{x}) - C_F(\bar{\lambda},\hat{x}) \\ & = f(x) + C_F(\bar{\lambda},x) - f(\tilde{x}) - C_F(\bar{\lambda},\tilde{x}) \\ & \geq \xi(x - \tilde{x}) \\ & = \xi(x - \hat{x}) + \xi(\hat{x} - \tilde{x}) \\ & = \xi(x - \hat{x}) \end{aligned}$$

That is:

$$L(\bar{\lambda},x) - L(\bar{\lambda},\hat{x}) \geq \xi(x - \hat{x}), \forall x \in X$$

So, $\xi\in\partial L(\bar{\lambda}, \hat{x})$.

Then:

$$\xi \in \partial L(\bar{\lambda},\hat{x}) \cap (-N_C(\hat{x}))$$

The conclusion follows since $\hat{x}$, $\tilde{x}$ is arbitrary.

Corollary 3.4: For problem, (P'), let $\bar{x}\in S'$, $\bar{\lambda}\in M'(\bar{x})$. Then:

$$\partial L'(\bar{\lambda},x) \cap (-N_C(x)) = \partial L'(\bar{\lambda},y) \cap (-N_C(y)), \forall x, y \in S'$$

That is, the function:

$$\partial L'(\bar{\lambda},\cdot) \cap (-N_C(\cdot))$$

Is constant on S'

Proposition 3.2: For problem (P), let $\bar{x}\in S$, $\bar{\lambda}\in M(\bar{x})$. then:

$$S = \{x \in E : C_F(\bar{\lambda},x) = 0, \partial L(\bar{\lambda},x) \cap (-N_C(x)) = \partial L(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x}))\}$$

Prove: Let:

$$S_2 = \{x \in E : C_F(\bar{\lambda},x) = 0, \partial L(\bar{\lambda},x) \cap (-N_C(x)) = \partial L(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x}))\}$$

From proposition 3.1, we have $S\subset S_2$. to prove $S_2\subset S$, let $x\subset S_2$, then:

$$C_F(\bar{\lambda}, x) = 0 \tag{3.7}$$

And:

$$\partial L(\bar{\lambda},x) \cap (-N_C(x)) = \partial L(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x}))$$

Since:

$$\partial L(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x})) \neq \varphi$$

Then:

$$\partial L(\bar{\lambda},x) \cap (-N_C(x)) \neq \varphi$$

Thus:

$$\partial L(\bar{\lambda},x) \cap (-N_C(x)) \neq \varphi$$

Together with (3.7), $x\in S$.

Corollary 3.5: For problem (P'), let $\bar{x}\in S'$, $\bar{\lambda}\in M'(\bar{x})$. Then:

$$S' = \{x \in E' : \prec \bar{\lambda}, g(x) \succ = 0, \partial L'(\bar{\lambda},x) \cap (-N_C(x)) = \partial L'(\bar{\lambda},\bar{x}) \cap (-N_C(\bar{x}))\}$$

Proposition 3.3: For problem (P), let $\bar{x}\in S'$, $\bar{\lambda}\in M'(\bar{x})$. Then:

$$S = \{x \in E : C_F(\bar{\lambda},x) = 0, 0 \in \partial L(\bar{\lambda},x) + N_C(x)\}$$

Prove: Let:

$$S_3 = \{x \in E : C_F(\bar{\lambda},x) = 0, 0 \in \partial L(\bar{\lambda},x) + N_C(x)\}$$

Obviously, we can obtain $S_3\subset S$ from Theorem 3.1 consequently, if $\bar{x}\in S$, By Proposition 3.2, we have, $\bar{x}\in S$.

Then, $C_F\ (\bar{\lambda}, x) = 0$ and there exist $\xi \in \partial L\ (\bar{\lambda}, x) \cap (-N_C\ (x))$, where S_2 is the set of Proposition 3.2. Hence, $x \in S_3$.

Corollary 3.6: for problem (P), let $\bar{x} \in S$, $\bar{\lambda} \in M\ (\bar{x})$. Then:

$$S' = \{x \in E' : \prec \bar{\lambda}, g(x) \succ = 0, 0 \in \partial L'(\bar{\lambda}, x) + N_C(x)\}$$

Note that, in the preceding corollary, the solution set S is characterized in terms of a fixed Lagrange multiplier $\bar{\lambda}$. Moreover, for any $x \in S$, the Lagrange multipliers are the same for each $x \in S$.

REFERENCES

Amahroq, T. and N. Gadhi, 2003. Second order optimality conditions for the extremely problem under inclusion constraints. J. Math. Anal. Appl., 285: 74-85.

Clarke, F.H., 1998. No smooth Analysis and Control Theory. Springer-Verlag, New York.

Dien, P.H., 1985. On the regularity condition for the extremely problem under locally lipschitz inclusion constraints. Appl. Math. Optim., 13: 151-161.

Ekland, I., 2009. Nonlinear Functional Analysis. Dover Publications Inc., New York.

Jeyakumar, V., G.M. Lee and N. Dinh, 2004. Lagrange multiplier conditions characterizing the optimal solution sets of Cone-constrained convex program. JOTA, 123: 83-103.