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A Note on Computing the Inverse and the Determinant of a Pentadiagonal Toeplitz Matrix in Signal Processing

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Abstract: Pentadiagonal Toeplitz systems of linear equations arise in many application areas and have been well studied over the past years, the invertibility of nonsingular pentadiagonal Toeplitz matrices has been quitely investigated in different fields of applied linear algebra. In this study, we provide a necessary and sufficient condition on which pentadiagonal Toeplitz matrix, present an algorithm for calculating the determinant of a pentadiagonal Toeplitz matrix and propose a fast algorithm for computing the inverse of a pentadiagonal Toeplitz matrix.

Key words: Lyapunov stability toeplitz matrix, pentadiagonal, fast algorithm, inverse, determinant

INTRODUCTION

Because of the structure and many good properties of pentadiagonal Toeplitz matrices, they have in recent years become one of the most important and active research field of applied mathematic and computation mathematic increasingly. They have been applied in many areas such as numerical solution of ordinary and partial differential equations, interpolation problems, boundary value problems and have a wide range of interesting applications as an important class of special matrices.

Consider the following n×n pentadiagonal matrix in this study:

$$\begin{bmatrix} b & a & 1 & & & & & \\ c & b & a & 1 & & & & \\ d & c & b & a & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & 1 \\ & & & & d & c & b & a \\ & & & & & d & c & b \end{bmatrix} \quad (1.1)$$

In (McNally *et al*., 2008; McNally, 2010) and (Nemani, 2010), the authors derived a new algorithm for solving symmetric pentadiagonal Toeplitz systems of linear equations based upon a technique used in (McNally *et al*., 2000) for tridiagonal Toeplitz systems. Based on the idea of a system perturbation followed by corrections, Nemani (Nemani, 2010) proposed a fast algorithm to solve the Toeplitz pentadiagonal system $Ax = f$. Jeffrey M.McNally, L.E.Garey, R.E. Shaw presented relevant background from these methods and then introduce an m processor scalable communication-less approximation algorithm for solving a diagonally dominant tridiagonal Toeplitz system of linear equations. In (Lv *et al*., 2008) presented an algorithm with the cost of 9n+3 for calculating the determinant of a pentadiagonal Toeplitz matrix and an algorithm for calculating the inverse of a pentadiagonal Toeplitz matrix.

Motivated by the above, in this study, we provide a necessary and sufficient condition on which pentadiagonal Toeplitz matrix, present an algorithm for calculating the determinant of a pentadiagonal Toeplitz matrix and propose a fast algorithm for computing the inverse of a pentadiagonal Toeplitz matrix.

In this study, $e_I = (0, \ldots, 1, 0, \ldots, 0)^T$, 1 at the i-th coordinate. A^T corresponds to the transpose matrix of A. Without loss of generality, we suppose $n \geq 11$.

PRELIMINARY NOTES

In this section, we present some lemmas that are important to our main results.

Lemma 2.1: Let matrix T be an n×n Toeplitz matrix:

$$A = T = \begin{bmatrix} 1 & & & & \\ t_1 & 1 & & & \\ t_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ t_{n-1} & \cdots & t_2 & t_1 & 1 \end{bmatrix}$$

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then the inverse of T be an n×n Toeplitz matrix also and:

$$T^{-1} = \begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_2 & a_1 & 1 \end{bmatrix}$$

where $\alpha_1 = t_1$:

$$a_j = -(t_j + \sum_{1<k<j} a_{k-1} t_{j-k})$$

$j \geq 2$.

Lemma 2.2: (Fan and Qian, 1994) Let:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is an n×n matrix, B is an n×m matrix, c is an m×n matrix, D is an m×m matrix. If A is invertible, then M is invertible if and only if $D\text{-}CA^{-1}B$ is invertible and:

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

MAIN RESULTS

In this study, without loss of generality, suppose that the pentadiagonal matrix A is nonsingular.

Decompose the pentadiagonal Toeplitz matrix A as the following perturbation:

$$A = MK^{n-2} \qquad (2.1)$$

$$K = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} M = \begin{bmatrix} 1 & & & & & & b & a \\ a & 1 & & & & & c & b \\ b & a & 1 & & & & d & c \\ c & b & a & 1 & & & 0 & d \\ d & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \\ & \ddots & c & b & a & 1 & 0 & \\ & & d & c & b & a & 0 & 0 \\ & & & d & c & b & 0 & 0 \end{bmatrix}$$

Let:

$$M = \begin{bmatrix} T & B \\ C & 0 \end{bmatrix}$$

where T is an (n-2)×(n-2) matrix, B is an (n-2)×2 matrix, c is an 2×(n-2) matrix and:

$$T = \begin{bmatrix} 1 & & & & & & & & \\ a & 1 & & & & & & & \\ b & a & 1 & & & & & & \\ c & b & a & 1 & & & & & \\ d & c & b & a & 1 & & & & \\ & d & c & b & a & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & d & c & b & a & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} b & a \\ c & b \\ d & c \\ & d \\ & \\ & \\ & \end{bmatrix}, C = B^T J, J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

It is easy to see T is invertible and by Lemma2.1:

$$T^{-1} = \begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & a_2 & a_1 & 1 \end{bmatrix}$$

Where:

$$\begin{aligned} &a_1 = -a,\ a_2 = -(b + aa_1) \\ &a_3 = -(c + ba_1 + aa_2), a_4 = -(d + ca_1 + ba_2 + aa_3) \\ &a_j = -(da_{j-4} + ca_{j-3} + ba_{j-2} + aa_{j-1}),\ j \geq 5 \end{aligned}$$

Let $N = B^T JT^{-1}B$. By above, it is easy to compute that:

$$N = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_1 \end{pmatrix}$$

Where:

$$\begin{aligned} &m_1 = a_n + aa_{n-1}, \\ &m_2 = a_{n-1}, \\ &m_3 = a^2 a_{n-1} + 2aa_n + a_{n+1} \end{aligned}$$

then M is invertible and:

$$M^{-1} = \begin{bmatrix} T^{-1} - T^{-1}BN^{-1}B^T JT^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^T JT^{-1} & -N^{-1} \end{bmatrix}$$

So:

$$A^{-1} = K^2 \begin{bmatrix} T^{-1} - T^{-1}BN^{-1}B^T JT^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^T JT^{-1} & -N^{-1} \end{bmatrix}$$

Thus, we have the following conclution:

Theorem Let A be a nonsingular pentadiagonal Toeplitz matrix and $A = MK^{n-2}$. Partition M as:

$$M = \begin{bmatrix} T & B \\ C & 0 \end{bmatrix}$$

where, M, T, B, C and J are as above. Then:

- A is invertible if and only if:

$$m_1^2 - m_2 m_3 \neq 0$$

- $\det A = (-1)^n (m_1^2 - m_2 m_3)$
- $$A^{-1} = K^2 \begin{bmatrix} T^{-1} - T^{-1}BN^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^T JT^{-1} & -N^{-1} \end{bmatrix}$$

Proof: Now we need to prove (2) only.

By the multiplication of block matrix, we have:

$$\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix} \begin{bmatrix} T & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} T & B \\ 0 & -CT^{-1}B \end{bmatrix}$$

So:

$$\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix} \begin{bmatrix} T & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} T & B \\ 0 & -CT^{-1}B \end{bmatrix}$$

Hence:

$$\det \begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix} \cdot \det \begin{bmatrix} T & B \\ C & 0 \end{bmatrix} = \det \begin{bmatrix} T & B \\ 0 & -CT^{-1}B \end{bmatrix}$$

Since:

$$\det \begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix} = 1, \det T = 1$$

So:

$$\det M = (-1)^n \det(B^T JT^{-1} B)$$

Finally, we have:

$$\text{DetA} = \det(MK^{n-2}) = \det M = (-1)^n (m_1^2 m_2 m_3)$$

The proof is complete.

According to the deduction above, we have the following algorithm:

Algorithm 1:

- **Step 1:** Using Lemma 2.1, calculate:

$$T^{-1} = \begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & a_2 & a_1 & 1 \end{bmatrix}$$

- **Step 2:** Calculate m_1, m_2, m_3
- **Step 3:** Calculate $\det A = (-1)^n (m_1^2 - m_2 m_3)$

Algorithm 2:

- Step 1 Using Lemma 2.1, calculate:

$$T^{-1} = \begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & a_2 & a_1 & 1 \end{bmatrix}$$

- **Step 2:** Calculate:

$$A^{-1} = K^2 \begin{bmatrix} T^{-1} - T^{-1}BN^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^T JT^{-1} & -N^{-1} \end{bmatrix}$$

NUMERICAL EXAMPLE

This section gives an example to illustrate our results.All the following tests are performed by MATLAB 7.0.

Example 1: Given $\alpha = 0$, b = 1, c = 0, d = 1 and n = 11, that is:

$$A = \begin{bmatrix} 1 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 1 & & & & \\ 1 & 0 & 1 & 0 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 & 0 & 1 \\ & & & & 1 & 0 & 1 & 0 \\ & & & & & 1 & 0 & 1 \end{bmatrix}_{11\times 11}$$

$$M = \begin{bmatrix} 1 & & & & & & 1 & 0 \\ 0 & 1 & & & & & 0 & 1 \\ 1 & 0 & 1 & & & & 1 & 0 \\ 0 & 1 & 0 & 1 & & & 0 & 1 \\ & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ & & 0 & 1 & 0 & 1 & 0 & \\ & & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$T=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\1&0&1&&&&&&\\0&1&0&1&&&&&\\1&0&1&0&1&&&&\\&1&0&1&0&1&&&\\&&1&0&1&0&1&&\\&&&1&0&1&0&1&\\&&&&1&0&1&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}1&0\\0&1\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}$$

So:

By Lemma 2.1, we have:

$$T^{-1}=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\-1&0&1&&&&&&\\0&-1&0&1&&&&&\\0&0&-1&0&1&&&&\\0&0&0&-1&0&1&&&\\1&0&0&0&-1&0&1&&\\0&1&0&0&0&-1&0&1&\\-1&0&1&0&0&0&-1&0&1\end{bmatrix}$$

$$a_9=0,\ a_{10}=0,\ a_{11}=0,\ a_{12}=1$$

$$N=\begin{pmatrix}m_1&m_2\\m_3&m_1\end{pmatrix}$$

Where:

$$m_3=a^2a_{10}+2aa_{11}+a_{12}=1$$
$$m_1=a_{11}+aa_{10}=0$$
$$m_2=a_{10}=0,\ m_1^2-m_2m_3=0$$

So, A is singular.

Example 2: Given $\alpha=0$, $b=2$, $c=0$, $d=1$ and $n=11$, that is:

$$A=\begin{bmatrix}2&0&1&&&&&&&&\\0&2&0&1&&&&&&&\\1&0&2&0&1&&&&&&\\&\ddots&\ddots&\ddots&\ddots&\ddots&&&&&\\&&\ddots&\ddots&\ddots&\ddots&\ddots&&&&\\&&&&&&1&0&2&0&1\\&&&&&&&1&0&2&0\\&&&&&&&&1&0&2\end{bmatrix}_{11\times 11}$$

So:

$$M=\begin{bmatrix}1&&&&&&2&0\\0&1&&&&&0&2\\2&0&1&&&&1&0\\0&2&0&1&&&0&1\\&\ddots&\ddots&\ddots&\ddots&\vdots&\vdots&\\&&0&2&0&1&0&\\&&1&0&2&0&0&0\\&&&1&0&2&0&0\end{bmatrix}$$

$$T=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\2&0&1&&&&&&\\0&2&0&1&&&&&\\1&0&2&0&1&&&&\\&1&0&2&0&1&&&\\&&1&0&2&0&1&&\\&&&1&0&2&0&1&\\&&&&1&0&2&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}2&0\\0&2\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}$$

By Lemma 2.1, we have:

$$T^{-1}=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\-2&0&1&&&&&&\\0&-2&0&1&&&&&\\3&0&-2&0&1&&&&\\0&3&0&-2&0&1&&&\\-4&0&3&0&-2&0&1&&\\0&-4&0&3&0&-2&0&1&\\5&0&-4&0&3&0&-2&0&1\end{bmatrix}$$

$$a_9 = 0,\ a_{10} = -6,\ a_{11} = 0,\ a_{12} = 7$$

$$N = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_1 \end{pmatrix}$$

Where:

$m_1 = \alpha_{11} + \alpha\alpha_{10} = 0$, $m_2 = \alpha_{10} = -6$, $m_2 = 7$

$$N = \begin{pmatrix} 0 & -6 \\ 7 & 0 \end{pmatrix}, N^{-1} = -\frac{1}{42}\begin{pmatrix} 0 & \frac{1}{7} \\ -\frac{1}{6} & 0 \end{pmatrix}$$

$$m_1^2 - m_2 m_3 = 42 \neq 0$$

A is invertible:

- det A = $(-1)^{11}\ m_1^2 - m_2 m_3 = -42$;
-

$$M^{-1} = \begin{bmatrix} -5/7 & 0 & 10/7 & 0 & -8/7 & 0 & 6/7 & 0 & -4/7 & 0 & 2/7 \\ 0 & -2/3 & 0 & 4/3 & 0 & -1 & 0 & 2/3 & 0 & -1/3 & 0 \\ 4/7 & 0 & -8/7 & 0 & 12/7 & 0 & -9/7 & 0 & 6/7 & 0 & -3/7 \\ 0 & 1/2 & 0 & -1 & 0 & 3/2 & 0 & -1 & 0 & 1/2 & 0 \\ -3/7 & 0 & 6/7 & 0 & -9/7 & 0 & 12/7 & 0 & -8/7 & 0 & 4/7 \\ 0 & -1/3 & 0 & 2/3 & 0 & -1 & 0 & 4/3 & 0 & -2/3 & 0 \\ 2/7 & 0 & -4/7 & 0 & 6/7 & 0 & -8/7 & 0 & 10/7 & 0 & -5/7 \\ 0 & 1/6 & 0 & -1/3 & 0 & 1/2 & 0 & -2/3 & 0 & 5/6 & 0 \\ -1/7 & 0 & 2/7 & 0 & -3/7 & 0 & 4/7 & 0 & -5/7 & 0 & 6/7 \\ 6/7 & 0 & -5/7 & 0 & 4/7 & 0 & -3/7 & 0 & 2/7 & 0 & -1/7 \\ 0 & 5/6 & 0 & -2/3 & 0 & 1/2 & 0 & -1/3 & 0 & 1/6 & 0 \end{bmatrix}$$

$$A^{-1} = K^2 \begin{bmatrix} T^{-1} - T^{-1}BN^{-1}B^TJT^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^TJT^{-1} & -N^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} 6/7 & 0 & -5/7 & 0 & 4/7 & 0 & -3/7 & 0 & 2/7 & 0 & -1/7 \\ 0 & 5/6 & 0 & -2/3 & 0 & 1/2 & 0 & -1/3 & 0 & 1/6 & 0 \\ -5/7 & 0 & 10/7 & 0 & -8/7 & 0 & 6/7 & 0 & -4/7 & 0 & 2/7 \\ 0 & -2/3 & 0 & 4/3 & 0 & -1 & 0 & 2/3 & 0 & -1/3 & 0 \\ 4/7 & 0 & -8/7 & 0 & 12/7 & 0 & -9/7 & 0 & 6/7 & 0 & -3/7 \\ 0 & 1/2 & 0 & -1 & 0 & 3/2 & 0 & -1 & 0 & 1/2 & 0 \\ -3/7 & 0 & 6/7 & 0 & -9/7 & 0 & 12/7 & 0 & -8/7 & 0 & 4/7 \\ 0 & -1/3 & 0 & 2/3 & 0 & -1 & 0 & 4/3 & 0 & -2/3 & 0 \\ 2/7 & 0 & -4/7 & 0 & 6/7 & 0 & -8/7 & 0 & 10/7 & 0 & -5/7 \\ 0 & 1/6 & 0 & -1/3 & 0 & 1/2 & 0 & -2/3 & 0 & 5/6 & 0 \\ -1/7 & 0 & 2/7 & 0 & -3/7 & 0 & 4/7 & 0 & -5/7 & 0 & 6/7 \end{bmatrix}$$

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