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Optimality Criterias for Nondifferentiable Multiobjective Fractional Programming Problems

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Abstract: The objective of this study is to study a class of nondifferentiable multiobjective fractional programming problems with inequality constraints where the objective and constraint functions are locally Lipschitz. By utilizing the assumptions of (b, α)-ρ-(η, θ)-invexity, Kuhn-Tucker type sufficient optimality conditions are obtained and proved for a feasible point to be a weakly (properly) efficient point with the help of the relationship between the single-objective optimization problem and multiobjective optimization problem. The results extend and improve the corresponding results in the literature.

Key words: (b, α)-ρ-(η, θ)-invexity, multiobjective fractional programming, sufficient optimality condition, efficient solution

INTRODUCTION

The multiobjective optimization problems are useful mathematical models for the investigation of real-world problems. As the generalization of single objective problem, multiobjective optimization problem enlarges the range of the application of optimization problems which arises in mechanical engineering, the design of aircraft control systems, resource planning and management, mathematical biology and so on (An and Gao, 2013). Many authors have obtained the necessary and sufficient optimality conditions for (weak) efficiency in such optimization problems. In particular, Jayswal considered a class of multiobjective programming problem under generalized α-type I univexity and established the sufficient optimality conditions and duality results. Necessary optimality conditions and duality results were obtained for a class of multiobjective semi-infinite programming by Gao (Gao, 2012a, 2012b, Jayswal, 2010). We can also refer to the works about multiobjective programming problem by Kanzi and Kim (Kanzi *et al*., 2010; Kim and Lee, 2010; An and Gao, 2013).

Furthermore, the studies about the optimizations of the multiobjective fractional programming problems have also been a focal issue due to in many practical optimization problems the objective functions are quotients of two functions. For example, several necessary and sufficient optimality conditions for a feasible point to be an efficient solution for nondifferentiable multiobjective fractional programming problems were obtained in the framework of generalized convexity by Kim and Chinchuluun,(Kim *et al.,* 2006; Chinchuluun *et al*., 2007). Recently, a large literature was developed around generalized convexity and its applications in multiobjective programming. Many authors investigated the optimality conditions and duality results for multiobjective (fractional) programming problems under the conditions of generalized convexity. By (X, α, ρ, d) convexity and (f, ρ)-convexity, Long and Liu presented the optimality conditions and duality results for fractional programming and the results have weakened the convexity hypothesis and made the important contribution in optimality theorems (Liu *et al*., 1998; Long, 2011). Then the readers are advised to consult other similar literatures by Mishra, Castellani (Gao, 2013; Mishra and Rautela, 2009; Castellani, 2001).

In this study, motivated by the above work, several sufficient optimality conditions are obtained for a class of multiobjective fractional programming problem under the assumptions of (b, α)-ρ-(η, θ)-invexity.

DEFINITIONS AND PRELIMINARIES

Definition 1: Let $X \subseteq R^n$. The function $f: X \to R$ is locally Lipschitz on X, if there exists a positive constant k, such that:

$$|f(x_1) - f(x_2)| \le k\|x_1 - x_2\|, \ \forall x_1, x_2 \in X \qquad (1)$$

Definition 2: If $f: X \to R$ is locally Lipschitz on X, the generalized Clarke directional derivative of f at $x \in X$ in the direction $d \in R^n$ is defined by:

$$f^0(x; d) = \lim_{(y,t)\to(x,0^+)} \sup \frac{f(y+td) - f(y)}{t} \qquad (2)$$

The generalized subgradient of a locally Lipschitz function f at $x\in X$ is defined by:

$$\partial f(x)=\{\xi\in R^n : f^0(x;d)\geq\langle\xi,d\rangle,\forall d\in R^n\} \tag{3}$$

Throughout the study, we will use the following definitions given by Gao, we always suppose that X is an open subset of R^n, $b:X\times X\times X[0,1]\to R^+$, $b(x_1,x_2)=\lim_{\lambda\to 0^+} b(x_1,x_2,\lambda)\geq 0, \alpha: X\times X\to R^+\setminus\{0\}$, $\eta: X\times X\to R^n$, $\theta: X\times X\to R^n$, $\rho\in R$.

Definition 3: A Lipschitz function $f:X\to R$ is said to be (b, α)-ρ-(η, θ) invex at $u\in X$, if there exists b, α, η, θ and ρ such that:

$$\begin{aligned}&b(x,u)[f(x)-f(u)]\\&\geq\langle\alpha(x,u)\xi,\eta(x,u)\rangle+\rho\|\theta(x,u)\|^2\end{aligned} \tag{4}$$

Remark 1: If in the above definition, we have strict inequality for any $x\neq u$, then we say that f is (b, α)-ρ-(η, θ)-strictly invex at $u\in X$.

Definition 4: A Lipschitz function $f: X\to R$ is said to be (b, α)-ρ-(η, θ)-pseudoinvex at $u\in X$, if there exists b, α, η, θ and ρ such that:

$$\begin{aligned}&\langle\alpha(x,u)\xi,\eta(x,u)\rangle+\rho\|\theta(x,u)\|^2\geq 0\\&\Rightarrow b(x,u)[f(x)-f(u)]\geq 0,\ \forall x\in X,\xi\in\partial f(u)\end{aligned} \tag{5}$$

Definition 5: A Lipschitz function $f:X\to R$ is said to be (b, α)-ρ-(η, θ)-quasiinvex at $u\in X$, if there exists b, α, η, θ and ρ, such that:

$$\begin{aligned}&b(x,u)[f(x)-f(u)]\leq 0\Rightarrow\\&\langle\alpha(x,u)\xi,\eta(x,u)\rangle+\rho\|\theta(x,u)\|^2\leq 0,\ \forall x\in X,\xi\in\partial f(u)\end{aligned} \tag{6}$$

Definition 6: A Lipschitz function $f:X\to R$ is said to be (b, α)-ρ-(η, θ)-strictly quasiinvex at $u\in X$, if there exists b, α, η, θ and ρ, such that:

$$\begin{aligned}&b(x,u)[f(x)-f(u)]\leq 0\\&\Rightarrow\langle\alpha(x,u)\xi,\eta(x,u)\rangle+\rho\|\theta(x,u)\|^2<0,\ \forall x\neq u\in X,\xi\in\partial f(u)\end{aligned} \tag{7}$$

OPTIMALITY CONDITIONS

Throughout this paper, let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant and X be a nonempty open subset of R^n. We use the following conventions for vectors in R^n:

$$x\leq y\Leftrightarrow x_i\leq y_i, x<y\Leftrightarrow x_i<y_i, \text{for } i=1,2,\cdots,n;$$
$$x\leq y\Leftrightarrow x_i\leq y_i, \text{for all } i=1,2,\cdots,n \text{ and } x\neq y;$$

$x\nless y$ is the negation of $x<y$.

In this section, we consider the optimality conditions for the following nondifferentiable multiobjective fractional programming problem in the framework of (b, α)-ρ-(η, θ)-invexity:

(MFP):

$$\min F(x)=\frac{f(x)}{g(x)}=(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\cdots,\frac{f_p(x)}{g_p(x)})$$
$$\text{s.t. } h(x)=(h_1(x),h_2(x),\cdots,h_m(x))?\ 0,\forall x\in X$$

where, X is an open subset of R^n, $f:X\to R$, $G:X\to R$, $h_j:X\to R$ (i = 1, 2,..., p; j = 1, 2,..., m) are real-valued locally Lipschitz functions. Without loss of generality, we assume that $f_i(x)\geq 0$, $g_i(x)>0$, I = 1, 2,..., p, for all $x\in X$. Let $X^0=\{x\in X|h(x)\leq 0\}$ be the feasible set of (MFP), let $I(x^0)=\{j\in\{1,2,\cdots,m\}|h_j(x^0)=0\}$ denote the index set of active constraints at given point $x^0\in X^0$.

Now, the following concepts of efficient solution for problem (MFP) are given.

Definition 7: A point $x^*\in X^0$ is said to be an efficient solution of (MFP), if there is no other $x\in X^0$, such that:

$$\frac{f_i(x)}{g_i(x)}?\ \frac{f_i(x^*)}{g_i(x^*)},\ \text{for all } i=1,2,\cdots,p \tag{8}$$

And:

$$\frac{f_i(x)}{g_i(x)}<\frac{f_i(x^*)}{g_i(x^*)},\ \text{for some } i\in\{1,2,\cdots,p\} \tag{9}$$

Definition 8: A point $x^*\in X^0$ is said to be a weakly efficient solution of (MFP), if there is no other $x\in X^0$, such that:

$$\frac{f_i(x)}{g_i(x)}<\frac{f_i(x^*)}{g_i(x^*)},\ \text{for all } i=1,2,\cdots,p \tag{10}$$

For:

$$\lambda_i>0(i=1,2,\cdots,p),\sum_{i=1}^{p}\lambda_i=1$$

we consider the following single-objective programming:

$$(MFP_\lambda)\quad \min G(x)=\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}$$
$$\text{s.t. } x\in X^0$$

Lemma 1: For a given:

$$\lambda_i > 0 (i = 1, 2, \cdots, p), \sum_{i=1}^{p} \lambda_i = 1$$

if $x^0 \in X^0$ is an optimal solution of (MFP_λ), then x^0 is a weakly efficient solution of (MFP).

Proof: Suppose contrary to the result that x^0 is not a weakly efficient solution of (MFP), then there exists $x \in X^0$, such that:

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^0)}{g_i(x^0)}, \text{ for all } i = 1, 2, \cdots, p \quad (11)$$

Since $\lambda_i > 0 (i = 1, 2, \ldots, p)$, the above inequality follows that:

$$\sum_{i=1}^{p} \lambda_i \frac{f_i(x)}{g_i(x)} < \sum_{i=1}^{p} \lambda_i \frac{f_i(x^0)}{g_i(x^0)} \quad (12)$$

Which is a contradiction to the optimality of x^0 for (MFP_λ).

Lemma 2: For a given:

$$\lambda_i > 0 (i = 1, 2, \cdots, p), \sum_{i=1}^{p} \lambda_i = 1$$

if $x^0 \in X^0$ is an optimal solution of (MFP_λ), then x^0 is a properly efficient solution of (MFP).

Proof: Suppose contrary to the result that x^0 is not a properly efficient solution of (MFP), then there exists one $i \in \{1, 2, \ldots, p\}$ and $x \in X^0$ satisfying $f_i(x) < f_i(x^0)$, such that for any $j \in \{1, 2, \ldots, p\}$ and $j \neq i$, we have:

$$f_j(x) < f_j(x^0), \quad (13)$$

And:

$$\frac{f_i(x^0) - f_i(x)}{f_j(x) - f_j(x^0)} > M \quad (14)$$

Now let:

$$M = (p-1) \max_{1 \le i,j \le p} \frac{\lambda_j}{\lambda_i} > 0$$

then the above inequality follows:

$$\begin{aligned} f_i(x^0) - f_i(x) &> M[f_j(x) - f_j(x^0)] \\ &= (p-1) \max_{1 \le i,j \le p} \frac{\lambda_j}{\lambda_i} [f_j(x) - f_j(x^0)] \\ &\ge \frac{p-1}{\lambda_i} \lambda_j [f_j(x) - f_j(x^0)] \end{aligned} \quad (15)$$

That is:

$$\frac{\lambda_i}{p-1} f_i(x^0) - f_i(x) > \lambda_j [f_j(x) - f_j(x^0)] \quad (16)$$

Summing the above inequality for any $j \neq i$, we get:

$$\lambda_i f_i(x^0) - f_i(x) > \sum_{j \neq i} [f_j(x) - f_j(x^0)] \quad (17)$$

Then the above inequality can be rewrited as follows:

$$\sum_{i=1}^{p} \lambda_i f_i(x^0) > \sum_{i=1}^{p} \lambda_i f_i(x) \quad (18)$$

Which is a contradiction to the optimality of $x^0$for (MFP_λ).

Hence, x^0 is a properly efficient solution of (MFP).

Theorem 1: Let $x^0 \in X^0$, suppose that:

- There exists:

$$\lambda_i > 0 \ (i = 1, 2, \cdots, p), \ \sum_{i=1}^{p} \lambda_i = 1$$

and $\mu_j \ge 0 (j = 1, 2, \ldots, m)$ such that:

$$0 \in \sum_{i=1}^{p} \lambda_i \partial(\frac{f_i}{g_i})(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$

$$\mu_j h_j(x^0) = 0, j = 1, 2, \cdots, m$$

- $\sum_{i=1}^{p} \lambda_i \frac{f_i}{g_i}$ is (b, α)-ρ-(η, θ)-invex and regular at x^0
- $\mu_j h_j$ is (c_j, α)-γ_j-(η, θ)-uasi- invex at x^0, j = 1, 2,..., m
- $b(x, x^0) > 0, c_j(x, x^0) \ge 0 (j = 1, 2, \cdots, m)$
- $\rho + \sum_{j=1}^{m} \gamma_j \ge 0$

Then x^0 is a weakly efficient solution of (MFP).

Proof: By the hypothesis (ii), for all $x \in X^0$, we have:

$$\begin{aligned}&b(x,x^0)[\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}-\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}\frac{f_i(x^0)}{g_i(x^0)}]\\&\geq\left\langle \alpha(x,x^0)\sum_{i=1}^{p}\lambda_i\xi_i,\eta(x,x^0)\right\rangle+\rho\left\|\theta(x,x^0)\right\|^2,\\&\forall\xi_i\in\partial(\frac{f_i}{g_i})(x^0)\end{aligned}\tag{19}$$

From the hypothesis (b), (iii) and $c_j(x, x^0)\geq 0$, we get:

$$\begin{aligned}&c_j(x,x^0)\left[\mu_j h_j(x)-\mu_j h_j(x^0)\right]?\ 0\\&\Rightarrow\left\langle \alpha(x,x^0)\mu_j\zeta_j,\eta(x,x^0)\right\rangle+\gamma_j\left\|\theta(x,x^0)\right\|^2\\&\leq 0,\forall\zeta_j\in\partial h_j(x^0),j=1,2,\cdots,m\end{aligned}\tag{20}$$

That is:

$$\left\langle \alpha(x,x^0)\sum_{j=1}^{m}\mu_j\zeta_j,\eta(x,x^0)\right\rangle+\sum_{j=1}^{m}\gamma_j\left\|\theta(x,x^0)\right\|^2?\ 0\tag{21}$$

According to Eq. 19 and 21, we obtain:

$$\begin{aligned}&b(x,x^0)[\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}-\sum_{i=1}^{p}\lambda_i\frac{f_i(x^0)}{g_i(x^0)}]\\&\geq\left\langle \alpha(x,x^0)(\sum_{i=1}^{p}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j),\eta(x,x^0)\right\rangle\\&+(\rho+\sum_{j=1}^{m}\gamma_j)\left\|\theta(x,x^0)\right\|^2\end{aligned}\tag{22}$$

By the hypothesis (a), there exists:

$$\forall\xi_i\in\partial(\frac{f_i}{g_i})(x^0)$$

and $\forall\zeta_j\in\partial h_j(x^0)$, such that:

$$\begin{aligned}&b(x,x^0)[\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}-\sum_{i=1}^{p}\lambda_i\frac{f_i(x^0)}{g_i(x^0)}]\\&\geq(\rho+\sum_{j=1}^{m}\gamma_j)\left\|\theta(x,x^0)\right\|^2\end{aligned}\tag{23}$$

Since $b(x, x^0)>0$, from the hypothesis (v), the above inequality follows that:

$$\sum_{i=1}^{p}\lambda_i\frac{f_i(x)}{g_i(x)}-\sum_{i=1}^{p}\lambda_i\frac{f_i(x^0)}{g_i(x^0)}\geq 0\tag{24}$$

Which implies that $x^0\in X^0$ is an optimal solution of (MFP_λ).

From Lemma 1, x^0 is a weakly efficient solution of (MFP).

Theorem 2: Let $x^0\in X^0$, suppose that:

- There exists:

$$\lambda_i>0\ (i=1,2,\cdots,p),\ \sum_{i=1}^{p}\lambda_i=1$$

and $\mu_j\geq 0(j = 1, 2,\ldots, m)$, such that:

$$0\in\sum_{i=1}^{p}\lambda_i\partial(\frac{f_i}{g_i})(x_0)+\sum_{j=1}^{m}\mu_j\partial h_j(x_0)$$

$$\mu_j h_j(x^0)=0,j=1,2,\cdots,m$$

- $\sum_{i=1}^{p}\lambda_i\frac{f_i}{g_i}$ is (b, α)-ρ(η, θ)-invex and regular at x^0
- h_j is (c_j, α)-γ_j(η, θ)-invex at x^0, j = 1, 2,…, m
- $b(x,x^0)>0,c_j(x,x^0)?\ 0(j=1,2,\cdots,m);$
- $\rho+\sum_{j=1}^{m}\gamma_j\geq 0$

Then x^0 is a weakly (properly) efficient solution of (MFP).

Proof: By the hypothesis (iii), for all $x\in X^0$, we have:

$$\begin{aligned}&c_j(x,x^0)\left[h_j(x)-h_j(x^0)\right]\\&\geq\left\langle \alpha(x,x^0)\zeta_j,\eta(x,x^0)\right\rangle+\gamma_j\left\|\theta(x,x^0)\right\|^2,\\&\forall\zeta_j\in\partial h_j(x^0),j=1,2,\cdots,m\end{aligned}\tag{25}$$

Since $\mu_j\geq 0$ and c_j(x, x0)≥0, it follows that:

$$\begin{aligned}&c_j(x,x^0)\left[\mu_j h_j(x)-\mu_j h_j(x^0)\right]\\&?\left\langle \alpha(x,x^0)\mu_j\zeta_j,\eta(x,x^0)\right\rangle+\mu_j\gamma_j\left\|\theta(x,x^0)\right\|^2\end{aligned}\tag{26}$$

According to the feasibility of x with (b) and (iv), we get:

$$\left\langle \alpha(x,x^0)\mu_j\zeta_j,\eta(x,x^0)\right\rangle+\mu_j\gamma_j\left\|\theta(x,x^0)\right\|^2\leq 0\tag{27}$$

That is:

$$\left\langle \alpha(x,x^0)\sum_{j=1}^{m}\mu_j\zeta_j,\eta(x,x^0)\right\rangle+\sum_{j=1}^{m}\mu_j\gamma_j\left\|\theta(x,x^0)\right\|^2\leq 0\tag{28}$$

This along with (a) and (v) yield there exists:

$$\forall \xi_i \in \partial(\frac{f_i}{g_i})(x^0)$$

such that:

$$-\left\langle \alpha(x, x^0)\sum_{i=1}^{p} \lambda_i \xi_i, \eta(x, x^0) \right\rangle - \rho \left\| \theta(x, x^0) \right\|^2 \le 0 \quad (29)$$

It implies:

$$\left\langle \alpha(x, x^0)\sum_{i=1}^{p} \lambda_i \xi_i, \eta(x, x^0) \right\rangle + \rho \left\| \theta(x, x^0) \right\|^2 \ge 0 \quad (30)$$

Using the hypothesis (ii), we have:

$$b(x, x^0)[\sum_{i=1}^{p} \lambda_i \frac{f_i(x)}{g_i(x)} - \sum_{i=1}^{p} \lambda_i \frac{f_i(x^0)}{g_i(x^0)}] \ge \left\langle \alpha(x, x^0)\sum_{i=1}^{p} \lambda_i \xi_i, \eta(x, x^0) \right\rangle + \rho \ge 0 \quad (31)$$

Since $b(x, x^0)>0$, it follows that:

$$\sum_{i=1}^{p} \lambda_i \frac{f_i(x)}{g_i(x)} - \sum_{i=1}^{p} \lambda_i \frac{f_i(x^0)}{g_i(x^0)} \ge 0 \quad (32)$$

Which implies that x^0 is an optimal solution of (MFP_λ).

From Lemma 1, x^0 is a weakly (properly) efficient solution of (MFP).

Theorem 3: Let $x^0 \in X^0$, suppose that:

- There exists:

$$\lambda_i > 0 \ (i = 1, 2, \cdots, p), \ \sum_{i=1}^{p} \lambda_i = 1$$

and $\mu_j \ge 0 (j = 1, 2, \ldots, m)$, such that:

$$0 \in \sum_{i=1}^{p} \lambda_i \partial(\frac{f_i}{g_i})(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$

$$\sum_{j=1}^{m} \mu_j h_j(x^0) = 0$$

- $\sum_{i=1}^{p} \lambda_i \frac{f_i}{g_i}$ is (b, α)-ρ-(η, θ)-pseudoinvex and regular at x^0, j = 1, 2,..., m
- h_j is (c_j, α)-γ_i-(η, θ)-invex at x^0, j = 1, 2,..., m
- $b(x, x^0) > 0, c(x, x^0) > 0$
- $\rho + \sum_{j=1}^{m} \mu_j \gamma_j \ge 0$

Then x^0 is a weakly (properly) efficient solution of (MFP).

Proof: The proof is similar to the proof of theorem 2.

CONCLUSION

Throughout this study, we have achieved and proved some sufficient optimality conditions for a kind of multiobjective fractional programming problem under the assumptions of (b, α)-ρ-(η, θ)-invexity. The results should be further opportunities for exploiting this structure of the multiobjective fractional programming problem.

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