http://ansinet.com/itj

ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 12 (23): 7401-7408, 2013
ISSN 1812-5638 / DOI: 10.3923/itj.2013.7401.7408

On Pseudo-spectral Method for Second-kind Weakly Singularvolterra Integral Equations with Smooth Solutions

[1]Zhang Xiao-Yong and [2]Wu Huafeng
[1]Shanghai Maritime University, China
[2]Merchant Marine College, Shanghai Maritime University, China

Abstract: The Jacobi pseudo-spectral Galerkin method for the weakly singular Volterra integral equations of the second kind with smooth solutions is proposed in this study. We provide a rigorous error analysis for the proposed method which indicates that the numerical errors (in the L^2-norm and the L^∞-norm) will decay exponentially provided that the source function is sufficiently smooth. Numerical examples are given to illustrate the theoretical results.

Key words: Volterra integral equation, Jacobi pseudo-spectral method, weakly singular kernel, convergence

INTRODUCTION

In practical applications one frequently encounters the Volterra integral equations of the second kind with a weakly singular kernel of the form:

$$\frac{dy}{dx} = b(x) + \int_0^x (x-s)^{-\mu} K(x,s)\, y(x)\, ds, \qquad (1)$$
$$0 < x \le T, 0 < \mu < 1$$

where, the unknown function y(x) is defined in $0<x\le T\le\infty$. b(x) is given source function and K(x, s) is a given kernel $y' \sim^{-\mu}$.

The numerical treatment of the Volterra integral Eq. 1 is not simple, mainly due to the fact that the solutions of Eq. 1 usually have a weak singularity at x = 0, As discussed in (Brunner, 1985, 2004; Brunner and Schotzau, 2006), the first derivative of the solution y(x) behaves like We point out that for Eq. 1 without the singular kernel (i.e., μ = 0) spectral methods and the corresponding error analysis have been provided recently for spectral methods to Volterra integral equations and pantograph-type delay differential equations. In both cases, the underlying solutions are smooth (Jiang, 2009; Shen and Tang, 2006).

In this study, we will consider a special case, namely, the exact solutions of Eq. 1 are smooth (Canuto *et al*., 2006). In this case, the collocation method and product integration method can be applied directly. But the main approach used there is the spectral-collocation method which is similar to a finite-difference approach. Consequently, the corresponding error analysis is more tedious as it does not fit in a unified framework. However, with a finite-element type approach, as will be performed in this work, it is natural to put the approximation scheme under the general Jacobi-Galerkin type framework. As demonstrated in the recent book of Shen *et al.* (2011), there is a unified theory with Jacobi polynomials to approximate numerical solutions for differential and integral equations. It is also rather straightforward to derive the pseudo-spectral Jacobi-Galerkin method from the corresponding continuous version. The relevant convergence theories under the unified framework, as will be seen from Sects.4, are cleaner and more reasonable than those obtained in Chen and Tang (2010).

The study is organized as follows. In section 2, we introduce the Jacobi pseudospectral Galerkin approaches for the Volterra integral Eq. 2. Some preliminaries and useful lemmas are provided in section 3. In section 4, the convergence analysis is given. We prove the error estimates in the L norm and L∞-norm. The numerical experiments are carried out in Section 5 which will be used to verify the theoretical results obtained in Section 4. The final section contains conclusions

SPECTRALMETHOD

Here, we formulate the Jacobi pseudo-spectral schemes for Eq. 1. For this purpose, Let $?_{\alpha,\beta} = (1-t)^\alpha (1+t)^\beta$ be a weight function in the usual sense, for $\alpha, \beta > -1, J_l^{\alpha,\beta}(t),\ l = 0,1,\ldots$ denote the Jacobi polynomials. The set of Jacobi polynomials $(J_l^{\alpha,\beta}(t))_{l=0}^\infty$ forms a orthogonal system.

For the sake of applying the theory of orthogonal polynomials, by the linear transformation:

$$x = \frac{T(1+t)}{2},\ s = \frac{T(1+t)}{2}$$

Corresponding Author: Zhang Xiao-Yong, Shanghai Maritime University, China

Letting:

$$(t)=y\left(\frac{T(1+t)}{2}\right), g(t)=b\left(\frac{T(1+t)}{2}\right),\ \Lambda=[-1,1]$$

the weakly singular Eq. 1 can be rewritten as follows:

$$u(t)=g(t)+\int_{-1}^{t}(t-s)^{-\mu}\tilde{K}(t,s)u(s)ds \tag{2}$$

Where:

$$\tilde{K}(t,s)=\left(\frac{T}{2}\right)^{1-\mu}K(t,s)$$

Next, we define a linear integral operator:

$$M\phi(t)=\int_{-1}^{t}(t-s)^{-\mu}\tilde{K}(t,s)\phi(s)ds=\int_{-1}^{1}(-\theta)^{-\mu}\left(\frac{t+1}{2}\right)^{1-\mu}\tilde{K}(t,\theta)\phi(\theta)d\theta \tag{3}$$

The Eq. 2 can be rewritten as:

$$u(t)=g(t)+Mu(t) \tag{4}$$

Now, let N be any positive integer and $P_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N. we denote the collocation points by$\{ti\}^{N}_{i=0}$ which is the set of (N+1) Jacobi Gauss point. We also define the Jacobi interpolating polynomial $I_N^{a,\beta}v\in P_N(\Lambda)$, satisfying $I_N^{a,\beta}v(t_i)=v(t_i),\ 0\le i\le N$.

It can be written as an expression of the form:

$$I_N^{a,\beta}v(t)=\sum_{i=0}^{N}v(t_i)F_i(t) \tag{5}$$

where, $F_i(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocationpoints $\{t_i\}_{i=0}^{N}$.

Now we describe the Jacobi pseudo-spectral method. Using (N +1)-point Gauss-Jacobi quadrature equation with weight $\omega_{-\mu,-\mu}$ to approximate Eq. 3 yields:

$$Mu(t)\cong M_N u(t)=\sum_{j=0}^{N}\left(\frac{t+1}{2}\right)^{1-\mu}\tilde{K}(\theta_j)\omega_{0\mu}(\theta_j)u(\theta_j)\omega_j \tag{6}$$

Instead of the continuous inner product, the discrete inner product will be implemented by the following equality:

$$(u,v)_N=\sum_{j=0}^{N}u(\theta_j)v(\theta_j)\omega_j \tag{7}$$

As a result, $(u,v)_{?_{-\mu,-\mu}}=(u,v)_N$ if $uv\in P_{2N}(\Lambda)$.
By the definition of $I_N^{-\mu,-\mu}$, we have:

$$(u,v)_N=(I_N^{-\mu,-\mu}u,v)_N \tag{8}$$

The Jacobi pseudo-spectral method is to find:

$$u_N(t)=\sum_{j=0}^{N}\tilde{u}_j J_j^{\mu,-\mu}(t)\in P_N(\Lambda)$$

such that:

$$(u_N,v)_N=M_N u_N+g,v)_N,\forall v\in P_N(?) \tag{9}$$

where, $\{\tilde{u}_j\}_{j=0}^{N}$ are determined by:

$$\sum_{j=0}^{N}\left\{(J_j^{-\mu-\mu},J_i^{-\mu-\mu})_N-(M_N J_j^{-\mu-\mu},J_i^{-\mu-\mu})_N\right\}\tilde{u}_j=\left(g,J_i^{-\mu-\mu}\right)_N \tag{10}$$

Denoting $\tilde{X}=[\tilde{u}_0,\tilde{u}_1,....,\tilde{u}_N]^T$, Eq. 11 yields a equation of the matrix form:

$$A\tilde{X}=g_N \tag{11}$$

Where:

$$A(i,j)=(J_j^{-\mu-\mu},J_i^{-\mu-\mu})_N-(M_N J_j^{-\mu-\mu},J_i^{-\mu-\mu})_N,$$
$$g_N(i)=(g_0,J_i^{-\mu-\mu})_N,0\le i\le N$$

SOME USEFULLEMMAS

We first introduce some Hilbert spaces. For simplicity, denote $\partial_t v(t)=(\partial/\partial_t)v(t)$, etc. For anonnegative integerm, define $H^m_{?_{a,\beta}}(-1,1):=\{v:\partial_t^k v(t)\in L^2_{?_{a,\beta}}(-1,1),0\le k\le m\}$, with the semi-norm and the norm as:

$$|v|_{L^2_{?_{a,\beta}}}=\partial_t^m v(t)_{L^2_{?_{a,\beta}}},v_m=(\sum_{k=0}^{m}\partial_t^k(t)^2_{L^2_{?_{a,\beta}}})^{\frac{1}{2}}$$

respectively. It is convenient sometime to introduce the semi-norms:

$$|v|_{L^2_{?_{a,\beta}}}=\partial_t^m v(t)_{L^2_{?_{a,\beta}}},v_m=(\sum_{k=0}^{m}\partial_t^k(t)^2_{L^2_{?_{a,\beta}}})^{\frac{1}{2}}$$

For bounding some approximation error of Jacobi polynomials, we need the following nonuniformly-weighted Sobolev spaces:

$$(u,v)_{m,*}=\sum_{k=0}^{m}(\partial_t^k u,\partial_t^k v)_{?_{a+k,\beta+k}},v_{m,*}=\sqrt{(v,v)_{m,*}}.$$

Next, we define the orthogonal projection $P_N: L^2(\Lambda) \to P_N(\Lambda)$ as $(u - P_N u, v) = 0, \forall v \in P_N(\Lambda)$.

P_N possesses the following approximation properties ((5.4.11), (5.4.12) and (5.4.24) on pp. 283-287 in Ref. (Lubich, 1985):

$$u - P_N u_{L^2(?)} \le cN^{-m} u_{H^m(?)} \tag{12}$$

And:

$$u - P_N u_{L^\infty} \le cN^{\frac{3}{4}-m} u_{m,\infty} \tag{13}$$

We have the following optimal error estimate for the interpolation polynomials based on the Jacobi Gauss points (Chen and Tang, 2010).

Lemma 1: For any function v satisfying $v \in H^m_{\omega_{\alpha,\beta},*}(-1,1)$ we have:

$$\|v - I_N^{a,\beta} v\|_{L^2_{?_{a,\beta}}(?)} \le cN^{-m} \partial_t^m v_{L^2_{?_{a+m,\beta+m}}} \tag{14}$$

for the Jacobi Gauss points and Jacobi Gauss-Radau points.

Lemma 2: If $v \in H^m_{?_{a,\beta},*}(-1,1)$, for some $m \ge 1$ and $\phi \in P_N(\Lambda)$, then for the Jacobi Gauss and Jacobi Gauss-Radau integration we have (Chen and Tang, 2010):

$$\begin{aligned} \left|(v,\phi)_{\omega_{\alpha,\beta}} - (v,\phi)_N\right| &\le v - I_N^{a,\beta} v_{L^2_{?_{a,\beta}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\le cN^{-m} \partial_t^m v_{L^2_{\omega_{\alpha+m,\beta+m}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}} \end{aligned} \tag{15}$$

Lemma 3: Let $\{F_j(t)\}_{j=0}^N$ be the N-thLagrange interpolation polynomials associatedwith the Gauss, or Gauss-Radau, or Gauss-Lobatto points of the Jacobi polynomials. Then:

$$I_N^{\alpha,\beta}{}_{L^\infty} := \max_{t\in[-1,1]} \sum_{j=0}^N |F_j(t)| = \begin{cases} c\log N, -1 < \alpha,\beta \le -\frac{1}{2} \\ cN^{\gamma+\frac{1}{2}}, \gamma = \max(\alpha,\beta), \textit{otherwise} \end{cases} \tag{16}$$

We now introduce some notation. For $r \ge 0$ and $\kappa \in [0,1], C^{r,?}([-1,1])$ will denote the space of functions whose r-th derivatives are Holder continuous with exponent k, endowed with the usual norm $\|.\|_{r,k}$. When k = 0,Cr,0 ([-1, 1]) denotes the space of functions withrcontinuous derivatives on [0, T], also denoted by C_r ([-1, 1]) and with norm$\|.\|_r$.

We will make use of a result of (Ragozin, 1970; Ragozin, 1971). Which states that, for each nonnegative integerr and k∈[0, 1], there exists a constant $C_{r,k}>0$ such that for any function $v \in C^{r,k}([-1,1])$ there existsa polynomial function $t_N v \in P_n$ such that:

$$v - t_N v_{L^\infty} \le C_{r,k} N^{-(r+k)} v_{r,k} \tag{17}$$

where, $\|.\|_\infty$ is the norm of the space $L^\infty([-1,1])$ and when the function $v \in C([-1,1])$. Actually, τ_N is a linear operator from $C^{r,k}([-1,1])$ to $P_N(?)$.

We will need the fact that M which be defined by Eq. 6, is compact as an operator from $C([0,T])$ to $C^{r,k}([-1,1])$ for any 0<k<1-μ (Chen and Tang, 2009).

We have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials;

Lemma 4: Let 0<k<1-μ then, for any function $v \in C([-1,1])$, there exists a positive constant C such that:

$$Mv_{0,k} \le Cv(t)_{L^\infty}, 0 < k < 1-\mu \tag{18}$$

In our analysis, we shall apply the generalization of Gronwalls Lemma. We call such a function v(t) locally integrable on the interval [0, T] if for each $t \in [0,T]$ its Lebesgue integral:

$$\int_0^t v(s)ds$$

is finite. The following result can be found in Federson *et al.* (2003).

Lemma 5: Suppose that:

$$v(t) \le w^*(t) + w(t)\int_0^t f(t,s)v(s)ds, t \in [0,T]$$

where, φw, φW and φV are locally integrable on the interval [0,T]. Here, all the functions are assumed to be nonnegative. Then:

$$v(t) \le w_*(t) + w(t)(e^{\int_0^t f(t,s)w(s)ds})\int_0^t f(t,s)w_*(s)ds$$

Lemma 6: Assume that v is a nonnegative, locally integrable function defined on [0, T] and satisfying:

$$v(t) \le w_*(t) + K_0 \int_0^t (t-s)^{-\mu} v(s)ds, t \in [0,T]$$

where, K_0 is a positive constant and w.(t) is a nonnegative and continuous function defined on [0,T].Then, there exists a constant C such that:

$$v(t) \le w_*(t) + C\int_0^t (t-s)^{-\mu} w_*(s)ds, t \in [0,T]$$

Lemma 7: Assume that v is a nonnegative, locally integrable function defined on [-1, 1] and satisfying:

$$v(t) \le w_*(t) + K_0 \int_{-1}^{t} (t-s)^{-\mu} v(s)ds$$

where, K_0 is a positive constant and w.(t) is a nonnegative and continuous function definedon [-1,1]. Then, there exists a constant C such that:

$$v(t) \le w_*(t) + C\int_{-1}^{t} (t-s)^{-\mu} w_*(s)ds$$

To prove the error estimate in the weighted L^2-norm, we need the generalized Hardys inequality with weights (Gogatishvill and Lang, 1999).

Lemma 8: For all measurable function f = 0, the following generalized Hardys inequality:

$$(\int_a^b |kf(x)|^q \omega_1(x)dx)^{\frac{1}{q}} \le c(\int_a^b |f(x)|^p \omega_2(x)dx)^{\frac{1}{p}}$$

holds if and only if:

$$\sup_{axb}(\int_x^b \omega_1(t)dt)^{\frac{1}{q}}(\int_a^x \omega_2^{1-q'}(t)dt)^{\frac{1}{q'}} < \infty$$

$$q' = \frac{p}{p-1}$$

for the case $1 \le p \le q < \infty$. Here, k is an operator of the form:

$$(kf)(x) = \int_a^x \rho(x,t)f(t)dt$$

with $\rho(x, t)$ a given kernel, ω_1, ω_2 areweight functions and $-\infty \le a < b \le \infty$.

We will need the following estimate for the Lagrange interpolation associated with the Jacobi Gaussian collocation points.

Lemma 9: For every bounded function v, there exists a constant C independent of v such that:

$$\|I_N^{\alpha,\beta} v(t)\|_{L^\infty} = \|\sum_{j=0}^{N} \left|v(t_j)F_j(t)\right|\|_{L^\infty} \le \begin{cases} c\, logN v_{L^\infty} & -1 < \alpha,\beta \le -\frac{1}{2} \\ cN^{\gamma+\frac{1}{2}} v_{L^\infty}, & \gamma = \max(\alpha,\beta), otherwise \end{cases}$$

where, F_j (t) is the Lagrange interpolation basis function associated with the collocation points $\{t_j\}^N_{j=0}$.

Lemma 10: For every bounded function v, there exists a constant C independent of v such that $I_N^{a,\beta} v(t)_{L^2_{\omega_{a,\beta}}} \le Cv_{L^\infty}$.

where, $F_j(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocationpoints. $\{t_j\}^N_{j=0}$

CONVERGENCE

As $I_N^{(-\mu,-\mu)}$ is the interpolation operator which is based on the (N+1)-degree Jacobi-Gauss pointswith weight $\omega_{-\mu,-\mu}$, in terms of Eq. 8-9, the pseudo-spectral solution u_N satisfies:

$$(u_N - I_N^{-\mu,-\mu} M_N u_N - I_N^{-\mu,-\mu} g, v_N)_{\omega-\mu,-\mu} = 0 \;\forall v_N \in P_N(\Lambda) \quad (19)$$

Where:

$$M_N u_N = M u_N - Q(t) \quad (20)$$

$$\begin{aligned} Q(t) &= \int_{-1}^{1} (\frac{t+1}{2})^{1-\mu} (1-?)^{-\mu} \bar{K}(t,?)u_N(?)d? \\ &-\sum_{j=0}^{N} (\frac{t+1}{2})^{1-\mu} \tilde{K}(s(t,?_j)))?_{0\mu}(?_j)u_N(s(t,?_j))?_j \\ &= ((\frac{t+1}{2})^{1-\mu} \tilde{K}(s(t,\bullet))?_{0,\mu}(\bullet)u_N(s(t,\bullet)))_{?-\mu,-\mu} \\ &-((\frac{t+1}{2})\tilde{K}(s(t,\bullet))?_{0\mu}(\cdot), u_N(s(t,\bullet)))_N \end{aligned} \quad (21)$$

in which $(.,.)_{\omega-\mu,-\mu}$ represents the continuous inner product with respect to θ and $(.,.)_N$ isthe corresponding discrete inner product defined by the Gauss-Jacobi quadrature formula. The combination of (4.1) and (4.2), yields:

$$(u_N + I_N^{-\mu,-\mu} Q(t) - I_N^{-\mu,-\mu} M u_N - I_N^{-\mu,-\mu} g, v_N)_{?_{-\mu,-\mu}} = 0$$

which gives rise to:

$$u_N + I_N^{-\mu,-\mu} Q(t) - I_N^{-\mu,-\mu} M u_N = I_N^{-\mu,-\mu} g \quad (22)$$

By the discussion above Eq. 17, 19 and 22 are equivalent.

Next we consider an auxiliary problem, we want to find $\hat{u}_N$, such that:

$$(\hat{u}_N, v_N)_N - (M\hat{u}_N, v_N)_N = (g, v_N)_N, \;\forall v_N \in ?_N(?) \quad (23)$$

where, M is the integral operator defined in Sect 2.In terms of the definition of $I_N^{-\mu,-\mu}$, Eq. 24 can be written as:

$$(\hat{u}_N, v_N)_N - (I_N^{-\mu,-\mu} M\hat{u}_N, v_N)_N = (I_N^{-\mu,-\mu} g, v_N)_N \;\forall v_N \in ?_N(?) \quad (24)$$

which is equivalent to:

$$\hat{u}_N - I_N^{-\mu,-\mu} M\hat{u}_N = I_N^{-\mu,-\mu} g \quad (25)$$

When g = 0, Eq. 25 can be written as $\hat{u}_N - I_N^{-\mu,-\mu} M\hat{u}_N = 0$.

In terms of the fact that:

$$\hat{u}_N - I_N^{\mu,-\mu}M\ \hat{u}_N = \hat{u}_N - M\ \hat{u}_N + \left(M\ \hat{u}_N - I_N^{\mu,-\mu}M\ \hat{u}_N\right)$$

Suppose that $\tilde{K}(t, s) \le L$. It is clear that from Eq. 2:

$$\left|\hat{u}_N\right| \le L\int_{-1}^{t}(t-s)^{-\mu}\left|\hat{u}_N(s)\right|ds + \left|I_N^{\mu,-\mu}M\ \hat{u}_N - M\ \hat{u}_N\right|$$

Using Lemma 7 leads to:

$$\|\hat{u}_{NL\infty} \le cI_N^{\mu,-\mu}M\ \hat{u}_N - M\,\hat{u}_{NL\infty} \quad (26)$$

We now estimate $\|\hat{u}_N\|_{L^\infty} \le c\|I_N^{\mu,-\mu}M\hat{u}_N - M\hat{u}_N\|_{L^\infty}$. By virtue of Eq. 14, Lemma 4 and 9,we obtain that:

$$\|I_N^{\mu,-\mu}M\hat{u}_N - M\hat{u}_N\|_{L^\infty} \le (1+\|I_N^{\mu,-\mu}\|_{L^\infty})M\hat{u}_N - t_N M\hat{u}_N\|_{L^\infty} \quad (27)$$

This, together with Eq. 26, gives:

$$\le \begin{cases} \log NN^{-?}\|\hat{u}_N\|_{L^\infty}, -1 < -\mu \le -\dfrac{1}{2}, \\ \le cN^{\frac{1}{2}-?-\mu}\|\hat{u}_N\|_{L^\infty}, \textit{Otherwise.} \end{cases}$$

which implies, when N is large enough and:

$$\mu + ? > \frac{1}{2},\ \hat{u}_N = 0$$

Hence, the $\hat{u}_N$ is existent andunique as $P_N(\Lambda)$ is finite-dimensional.

Lemma 1: Suppose that $u \in H^m_{?-\mu,-\mu}(?)$ and $\hat{K}(t,s) \le L$ then we have:

$$\|u - \hat{u}_N\|_{L^\infty(\Lambda)} \le \begin{cases} c\log NN^{\frac{3}{4}-m}\left(\|u\|_{m,\infty}\right), -1 < -\mu \le \dfrac{1}{2} \\ cN^{\frac{5}{4}-m-\mu}\left(\|u\|_{m,\infty}\right), \textit{Otherwise.} \end{cases} \quad (28)$$

and:

$$\|u - u_N\|_{L^2_{?-\mu,-\mu}(?)} \le \begin{cases} c\log NN^{\frac{3}{4}-m}\|u\|_{m,\infty} + cN^{-m}\partial_t^m\|u\|_{L^2_{?m-\mu,m-\mu}} \\ cN^{\frac{5}{4}-m-\mu}\|u\|_{m,\infty} + cN^{-m}\partial_t^m\|u\|_{L^2_{?m-\mu,m-\mu}}, \textit{otherwise} \end{cases} \quad (29)$$

Proof: Subtracting Eq. 25 from 4 yields:

$$u(t) - \hat{u}_N + I_N^{\mu,-\mu}M\hat{u}_N - Mu(t) = g(t) - I_N^{\mu,-\mu}g \quad (30)$$

Setting $\varepsilon u(t)$-$\hat{u}_N$ yields:

$$e = u - I_N^{\mu,-\mu}u + Me - \left[Me - I_N^{\mu,-\mu}Me\right] \quad (31)$$

which implies that:

$$|e| \le |J_1| + |J_2| + L\int_{-1}^{t}(t-s)^{-\mu}|e(s)|ds \quad (32)$$

where, $J_1 = u - I_N^{\mu,-\mu}u$, $J_2 = Me - I_N^{\mu,-\mu}Me$. Using Lemma 7 gives:

$$|e| \le |J_1| + |J_2| + c\int_{-1}^{t}(t-s)^{-\mu}(|J_1| + |J_2|)ds \quad (33)$$

Then, it follows from Eq. 33 that:

$$\|e\|_{L^\infty} \le c(\|J_1\|_{L^\infty} + \|J_2\|_{L^\infty}) \quad (34)$$

By using Eq. 20, Lemma 9, we obtain that:

$$|u - I_N^{\mu,-\mu}u\|_{L^\infty} \le c(1+\|I_N^{\mu,-\mu}\|_{L^\infty})\,\|u - P_N u\|_{L^\infty}$$
$$\le \begin{cases} c\log NN^{\frac{3}{4}-m}u_{m,\infty}, -1 < -\mu \le -\dfrac{1}{2}, \\ cN^{\frac{5}{4}-m-\mu}u_{m,\infty}, -\dfrac{1}{2} < -\mu < 0. \end{cases} \quad (35)$$

We now estimate the third term l_2. It is clear that $\varepsilon\epsilon c[-1, 1]$. Consequently, using Eq. 24 and Lemma 22 it follows that:

$$\|I_2\|_{L^\infty} = \|\left(I - I_N^{\mu,-\mu}\right)(Me - t_N Me)\|_{L^\infty} \le c(1+\|I - I_N^{\mu,-\mu}\|_{L^\infty})$$
$$Me - t_N Me\|_{L^\infty} \le \begin{cases} c\log NN^{-k}e_{L^\infty}, -1 < -\mu \le -\dfrac{1}{2} \\ cN^{\frac{1}{2}-k-\mu}e_{L^\infty}, \textit{otherwise} \end{cases} \quad (36)$$

where, $\kappa\epsilon(0, 1-\mu)$ and $\tau_N M\varepsilon\epsilon P_N(\Lambda)$. The estimate Eq. 38 follows from Eq. 34-36, provider that N is large enough and:

$$? + \mu > \frac{1}{2}$$

Next we prove Eq. 29. Using the generalized Gronwall inequality (Lemma 8), we have from Eq. 32 that:

$$\begin{aligned} \|e\|^2_{L^2_{?-\mu\mu}} &\le C(\|I_1\|^2_{L^2_{?-\mu\mu}} + \|I_2\|^2_{L^2_{?-\mu\mu}} + \|e\|^2_{L^2}) \\ &\le C(\|I_1\|^2_{L^2_{?-\mu\mu}} + \|I_2\|^2_{L^2_{?-\mu\mu}} + \|e\|^2_{L^\infty}) \end{aligned} \quad (37)$$

From Lemma 10, we obtain that:

$$\|I_1\|^2_{L^2_{?-\mu\mu}} \le c\|e - t_N Me\| \le cN^{-?}\|e\|_{L^\infty}$$

This result, together with the estimates Eq. 28, 37 and 21 yields Eq. 39.

Now setting $E = \hat{u}_N - u_N$, subtracting Eq. 32 from 35 leads to:

$$E - I_N^{\mu,-\mu}Q(t) - I_N^{\mu,-\mu}ME = 0 \tag{38}$$

Let $e_N = u - y_N$ be the Jacobi pseudo-spectral solution u_N of Eq. 16. Now we are prepared to get our global convergence result for problem Eq. 2.

Theorem 1: Suppose that $|\tilde{K}(t,s)| \le L$ and the solution of Eq. 2 is sufficiently smooth. For the Jacobi pseudo spectral solution defined in Eq. 6, we have the following estimates:

- L^∞ norm of $|e_N|$ satisfies:

$$\|| e_N \||_{L^\infty} \le \begin{cases} c \log N(?^{-m}\|u\|_\infty + ?^{\frac{3}{4}-m}\|u\|_{m,\infty}))(herwise, -1 < -\mu \le \frac{1}{2} \\ c\left(?^{\frac{1}{2}-m-\mu}\|u\|_\infty + ?^{\frac{5}{4}-m-\mu}\|u\|_{m,\infty}\right), \ otherwise \end{cases}$$

- $L^2_{\omega\text{-}\mu,\text{-}\mu}$ norm of $|e_N|$ satisfies:

$$\|e_N\|_{L^2_{?-\mu,-\mu}} \le \begin{cases} c \log N?^{-m} \| (|u|) \|_{L^\infty} + c \log N N^{\frac{3}{4}-m} \|u\|_{m,\infty} \\ \quad + cN^{-m} \| \partial_t^m u \|_{?_{m-\mu,m-\mu}}, -1 < -\mu \le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m-\mu} \|u\|_{L^\infty} + cN^{\frac{5}{4}-m-\mu}\left(\|u\|_{m,\infty}\right) \\ \quad + cN^{-m}\left(\|\partial_t^m u\|_{?_{m-\mu,m-\mu}}\right), -\frac{1}{2} < -\mu \le 0. \end{cases} \tag{39}$$

Proof: We prove the existence and uniqueness of the Jacobi pseudo-spectral solution u_N. As the dimension of $p_N(\Lambda)$ is finite and Eq. 6 and Eq. 32 are equivalent, we only need to prove that the solution of Eq. 32 is $u_N = 0$ when $g = 0$. We consider the equation:

$$u_N + I_N^{\mu,-\mu}Q(t) - I_N^{\mu,-\mu}?\ u_N = 0 \tag{40}$$

Obviously Eq. 40 can be written as:

$$u_N = \int_{-1}^{t}(t-s)^{-\mu}\ \hat{K}(t,s)u_N(s)ds + J_1 + J_2 \tag{41}$$

$J_1 = I_N^{\mu,-\mu}?\ u_N - ?\ u_N,\ J_2 = -I_N^{\mu,-\mu}Q(t)$. Using Eq. 41 gives:

$$|u_N| \le |J_1| + |J_2| + L\int_{-1}^{t}(t-s)^{-\mu}|u_N(s)|ds \tag{42}$$

Using Lemma 7 yields:

$$\|u\|_{N L^\infty} \le c\left(\|J_1\|_{L^\infty} + \|J_2\|_{L^\infty}\right) \tag{43}$$

On the other hand, according to Lemma 9:

$$\|J_2\|^2_{L^\infty} = \|I_N^{\mu,-\mu}Q(t)\|^2_{L^\infty} \le \begin{cases} c(\log N)^2 \|Q(t)\|^2_{L^\infty}, -1 < -\mu \le -\frac{1}{2}, \\ cN^{1-2\mu}\|Q(t)\|^2_{L^\infty}, \ otherwise. \end{cases}$$

By the expression of Q(t) in Eq. 31 and Lemma 2, we have $|Q(t)| \le cN^{-m}\|u\|_{N L^2_{?-\mu,-\mu}}$ which, together with Eq. 43, gives:

$$\|J_2\|_{L^\infty} \le \begin{cases} c \log N N^{-m}\|u_N\|_{L^\infty}, -1 < \mu \le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m-\mu}\|u_N\|_{L^\infty}, otherwise. \end{cases} \tag{44}$$

The combination of Eq. 34, 42 and 44 yields:

$$\|u\|_{L^\infty} \le \begin{cases} c \log N(N^{-?-\mu} + N^{-m-\mu})\|u_N\|_{L^\infty}, -1 < \mu \le -\frac{1}{2}, \\ c(N^{\frac{1}{2}-?-\mu} + N^{\frac{1}{2}-m-\mu})\|u_N\|_{L^\infty}, otherwise. \end{cases} \tag{45}$$

Based on Eq. 45 and Lemma 4 with:

$$\kappa + \mu > \frac{1}{2}$$

when N is largeenough $u_N = 0$. As a result, the existence and uniqueness of the pseudo-spectral solution u_N isproved. Now we turn to the error estimate of u-u_N. Actually Eq. 35 can be transformed into:

$$|E| \le L\int_{-1}^{t}(t-s)^{-\mu}|E(s)|ds + |J_3| + |J_4| \tag{36}$$

with $J_3 = I_N^{\mu,-\mu}Q(t), J_4 = I_N^{\mu,-\mu}ME - ME$. It follows from Eq. 36 and Lemma 7 that:

$$|E| \le c\int_{-1}^{t}(t-s)^{-\mu}(|J_3| + |J_4|)ds + |J_3| + |J_4| \tag{37}$$

which yield s:

$$E_{L^\infty} \le c\left(J_{3L^\infty} + J_{4L^\infty}\right) \tag{38}$$

Similar to the estimate of Eq. 27. We obtain:

$$\|J_4\|_{L^\infty} \le \begin{cases} c \log N N^{-\kappa}\|E\|_{L^\infty}, -1 < -\mu \le \frac{1}{2} \\ cN^{\frac{1}{2}-\kappa-\mu}\|E\|_{L^\infty}, \text{otherwise}. \end{cases} \tag{39}$$

By Eq. 37, we have that:

$$\|J_3\|_{L^\infty} \le \begin{cases} c\log N N^{-m}\|u_N\|_{L^\infty}, -1<-\mu\le \frac{1}{2} \\ cN^{\frac{1}{2}-m-\mu}\|u_N\|_{L^\infty}, \text{otherwise.} \end{cases} \tag{40}$$

In terms of Eq. 38-40, we obtain:

$$\|E\|_{L^\infty} \le \begin{cases} c\log N N^{-m}(\|u-u_N\|_{L^\infty}+\|u\|_{L^\infty})-1<-\mu\le \frac{1}{2} \\ cN^{\frac{1}{2}-m-\mu}\left(\|u-u_N\|_{L^\infty}+\|u\|_{L^\infty}\right), \textit{otherwise} \end{cases} \tag{41}$$

By the triangular inequality:

$$\|u-u_N\|_{L^\infty} \le \|u-\hat{u}_N\|_{L^\infty} + \|\hat{u}_N-u_N\|_{L^\infty} = \|u-\hat{u}_N\|_{L^\infty} + E_{L^\infty} \tag{42}$$

as well as Eq. 40-41 and Lemma 4.1, we can obtain the estimated Eq. 39 provided N is sufficiently large. Next we prove Eq. 40. Using the generalized Hardy inequality(Lemma 8, p = q = 2), one obtains that from (4.31):

$$\|E\|^2_{L^2_{w^{-\mu,-\mu}}} \le c\left(\|J_3\|^2_{L^2_{?^{-\mu,-\mu}}} + \|J_4\|^2_{L^2_{?^{-\mu,-\mu}}}\right) \le c\left(\|J_3\|^2_{L^\infty} + \|J_4\|^2_{L^\infty}\right) \tag{43}$$

The combination of (4.33), (4.34) and (4.37) yields:

$$\|E\|^2_{L^2_{w^{-\mu,-\mu}}} \le \begin{cases} c\log N(NN^{-?}\|E\|_{L^\infty} + N^{-m}\|u_N\|_{L^\infty}), -1<-\mu\le\frac{1}{2}, \\ c\left(N^{\frac{1}{2}-?-\mu}\|E\|_{L^\infty} + N^{\frac{1}{2}-m-\mu}\|u_N\|_{L^\infty}\right), \textit{ otherwise} \end{cases} \tag{44}$$

By the triangular inequality again:

$$\|u-u_N\|_{L^2_{?^{-\mu,-\mu}}} \le \|u-\hat{u}_N\|_{L^2_{?^{-\mu,-\mu}}} + \|E\|_{L^2_{?^{-\mu,-\mu}}}$$

In terms of (4.35)(4.38)(4.39) and Lemma 4.1, we obtain the desired result.

NUMERICAL RESULTS

We give two numerical example to confirm our analysis.

Example 1: Consider the second-kind weakly singular Volterra integral equation:

$$u(t) = e^{2t} + \frac{4(t+1)^{\frac{3}{4}}}{3} - \frac{16(t+1)^{\frac{7}{4}}}{21} + \int_{-1}^{t}(t-t)^{-\frac{1}{4}}\, t e^{-2t} u(t)dt$$

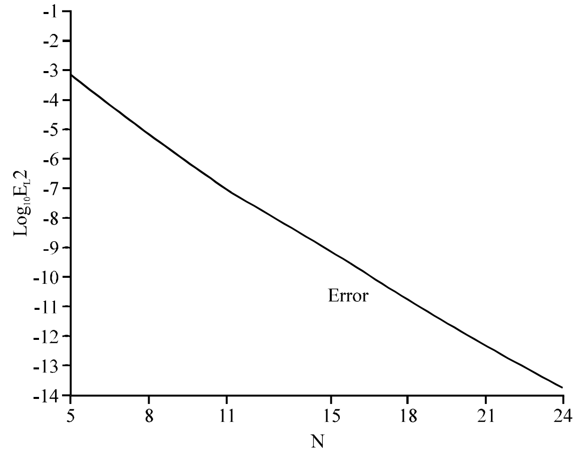

Fig. 1: $L^2_{\omega_{\alpha\beta}}$ error of example 6.1

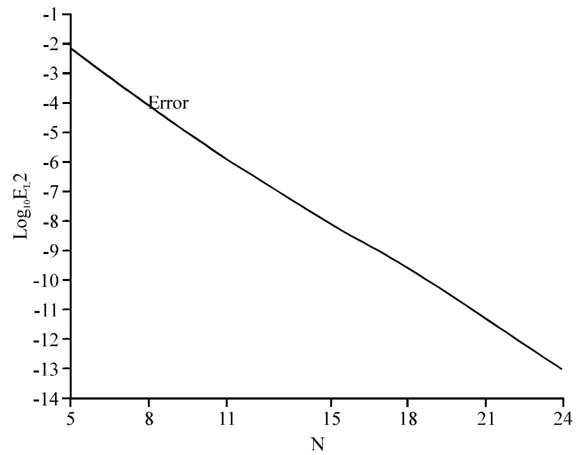

Fig. 2: ^{2} error of example 6.2

The exact solution is $u(t) = -e^{2t}$ Fig. 1 shows the errors u-u_N of approximate solution inweighted $L^2_{\omega_{\alpha,\beta}}$ and Fig. 3 shows the errors L^∞ norms obtained by using the pseudo-spectral methods described above. It is observed that the desired exponential rate of convergence is obtained.

Example 2: Consider the second-kind weakly singular Volterra integral equation:

$$u(t) = \cos 0.5t - \frac{16(t+1)^{\frac{5}{4}}}{5} + 4(t+1)^{\frac{7}{4}} + \int_{-1}^{t}(t-t)^{-\frac{3}{4}}\frac{t}{\cos 0.5t}u(t)dt$$

The exact solution is given by $u(t) = \cos 0.5t$.

Figure 2 and 4 plot the errors u-u_N for $5\le N\le 24$ in $L^2_{\omega_{\alpha,\beta}}$ and L^∞ norms. Onceagain the desired spectral accuracy is obtained. In the Fig. 3-4, $\| e_N \|_{L^\infty}$ is the maximumpoint-wise error, i.e.:

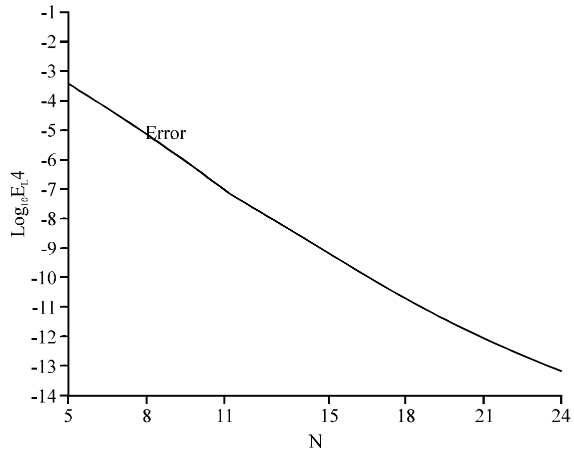

Fig. 3: L^∞ error of example 6.1

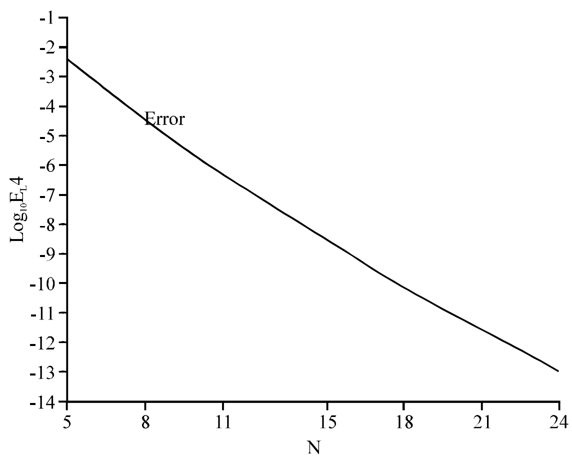

Fig. 4: L^∞ error of example 6.2

$$\| e_N \|_{L^\infty} = \max_{0 \le j \le N} e_N(t_j)$$

ACKNOWLEDGMENT

This study was financially supported by the National Natural Science Foundation of China (51279099), Shanghai Natural Science Foundation (12ZR1412500), Innovation Program of Shanghai Municipal Education Commission (13ZZ124).

REFERENCES

Brunner, H. and D. Schotzau, 2006. Hp-Discontinuous galerkin time-stepping for volterra integrodifferential equations. SIAM J. Numer. Anal., 44: 224-245.

Brunner, H., 1985. The numerical solution of weakly singular volterra integral equations by collocation on graded meshes. Math. Comput., 45: 417-437.

Brunner, H., 2004. Collocation Methods for Volterra Integral and Related Functional Differential Equations. Cambridge University Press, Cambridge, UK.

Canuto, C., M.Y. Hussaini, A. Quarteroni and T.A. Zang, 2006. Spectral Methods Fundamentals in Single Domains. Springer-Verlag, Berlin, Germany.

Chen, Y.P. and T. Tang, 2009. Spectral methods for weakly singular Volterra integral equations with smooth solutions. J. Computat. Applied Math., 233: 938-950.

Chen, Y.P. and T. Tang, 2010. Convergence analysis of the Jacobi Spectral-Collocation methods for Volterra integral equations with a weakly singular kernel. Math. Comput., 79: 147-167.

Federson, M., R. Bianconi and L. Barbanti, 2003. Linear Volterra integral equations as the limit of discrete systems. Cadernos Matematica, 4: 331-352.

Gogatishvill, A. and J. Lang, 1999. The generalized hardy operator with kernel and variable integral limits in banach function spaces. J. Inequalities Appl., Vol. 4. 10.1155/S1025583499000272

Jiang, Y.J., 2009. On spectral methods for Volterra-type Integro-differential equations. J. Comput, Appl. Math., 230: 333-340.

Lubich, C., 1985. Fractional linear multistep methods for Abel-Volterra integral equations of the second kind. Math. Comput., 45: 463-469.

Ragozin, D.L., 1970. Polynomial approximation on compact manifolds and homogeneous spaces. Trans. Am. Math. Soc., 150: 41-53.

Ragozin, D.L., 1971. Constructive polynomial approximation on spheres and projective spaces. Trans. Am. Math. Soc., 162: 157-170.

Shen, J. and T. Tang, 2006. Spectral and High-Order Methods with Applications. Science Press, Beijing, China, ISBN: 9787030177223, Pages: 326.

Shen, J., T. Tang and L.L. Wang, 2011. Spectral Methods: Algorithms, Analysis and Applications. Springer, New York, USA., ISBN: 9783540710417, Pages: 486.