http://ansinet.com/itj

ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 12 (23): 7815-7820, 2013
ISSN 1812-5638 / DOI: 10.3923/itj.2013.7815.7820

Novel Criteria of Synchronization Stability in Complex Networks with Time-varying Inner-coupling Function and Time Delays

Jianeng Tang
College of Engineering, Huaqiao University, Quanzhou, 362021, Fujian, Pepole's Republic of China

Abstract: In this study, we further study the complex dynamical networks with time-varying inner-coupling functions and time delays. Based on the theory of asymptotic stability of linear time-delay systems, several synchronization criteria are established for such network. As illustrative examples, we use the networks with coupling delays and a given coupling scheme to test the theoretical results.

Key words: Complex network, synchronization stability, linear time-delay systems, time delay

INTRODUCTION

Complex networks are the sets of interconnected large-scale nodes, in which a node is a fundamental unit that can have different meanings in different situations, for example, chemical substrates, microprocessors, computers, schools, companies, papers, webs and people. Examples of all kinds of complex networks contain the Internet, the World Wide Web, food webs, electric power grids, cellular and metabolic networks, etc. (Strogatz, 2001). These good-sized complex networks always show better cooperative or synchronous behaviors among their constituents.

Complex networks were conventionally researched by graph theory, for which a complex network was described by a random graph, where the radical theory was introduced by Erdös and Rényi (Erdos and Renyi, 1960). Watts and Strogatz (WS) (Watts and Strogatz, 1998) introduced the conception of small-world networks to describe a transition from a regular lattice to a random graph. Another significant discovery in the field of complex networks is the observation that many large-scale complex networks are essentially scale free (Barabasi and Albert, 1999). After these networks were introduced, much attention has been paid to them to explore their complexity (Albert and Barabasi, 2002).

As a matter of fact, time delays commonly exist in the world. The dynamics of networks with delayed coupling have been extensively studied in recent years. A model of dynamical networks with multi-links was constructed in Refs. (Peng *et al*., 2010). In the study (Hoang, 2011), the complex synchronization manifold was demonstrated for the first time which was generated in coupled multiple time delay systems. Importantly, the synchronization of complex networks with delays still provides several scientific challenges and some new stability criteria are yet to be discovered. In the present study, we further study the synchronization of complex dynamical networks with time-varying inner-coupling functions and time delays. Based on the theory of asymptotic stability of linear time-delay systems, the stability criteria of the synchronization state are derived.

CRITERIA OF SYNCHRONIZATION

Here, based on the theory of asymptotic stability of linear time-delay systems, we will develop novel criteria of synchronization stability in complex networks with time-varying inner-coupling function and time delays. Firstly, we will make some preliminaries as follows.

Consider a time-delay dynamical system:

$$\dot{x} = A(t)x + B(t)x(t-\tau) \tag{1}$$

where, $x \in R^N$, $A(t)$, $B(t) \in R^{n\times n}$, $B(t)$ is an integral function matrix and $\tau > 0$ is time delay. The fundamental results which give the condition for stability or asymptotic stability of system 1 are summarized in the following lemmas, respectively.

Lemma 1: Vidyasagar (1993). Let A (t) be a constant matrix. Stability of zero solution of system 1 is achieved if the following conditions are satisfied:

$$\dot{x} = Ax \tag{2}$$

- Zero solution of system 2 is stable:
- $\int_{t_0}^{\infty} \|B(t)\| d_t < \infty$

Lemma 2: Vidyasagar (1993). Let A (t) be a continuous periodic function matrix. Asymptotical stability of zero solution of system 1 is achieved if the following conditions are satisfied:

$$\dot{x} = A(t)x, A(t+\omega) = A(t) \tag{3}$$

- Zero solution of system 3 is asymptotically stable
- $\int_{t_0}^{\infty} \|B(t)\| d_t < \infty$

||X|| denotes the norm of the matrix X.

The above two lemmas which include information of the delay is referred to as the stability criterion. It is noticed that condition 1 is the stable problem of the ordinary differential equation. Hence, we resort to the following criterion to explore the stability of the time-delay system.

Consider an ordinary differential dynamical system:

$$\dot{x} = A(t)x \tag{4}$$

where, $x = (x_1, x_2, \ldots, x_n)^T$. Let $D = (d_{ij}(t))_{n\times n}$, $A = (a_{ij}(t))_{n\times n}$ and $d_{ij(t)} = (a_{ij}(t)+a_{ji}(t))/2$, i, j = 1, 2,…, n.

Lemma 3: Hirsch *et al*. (2008). If D is a negative definite matrix and an often negative matrix, then zero solution of system 4 is stable; If D is a positive definite matrix, then zero solution of system 4 is unstable; If D is a negative definite matrix, then zero solution of system 4 is asymptotically stable.

In what follows, we consider a general complex dynamical network consisting of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system and introduce the coupling delays and time-varying inner-coupling function in this network. This dynamical network is described by:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} C_{ij} G(t)(x_j(t-\tau)) \tag{5}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \epsilon R^n$ is a state vector representing the state variables of node i, i = 1, 2,…,N. $G(t) = (a_{ij}(t)) \epsilon R^{n\times n}$ is a coupling link matrix between node i and node j (i ≠ j) for all 1 ≤i, j≤N at time t, the constant c>0 is the coupling strength, $C(t) = (C_{ij})_{N¡ÁN}$ is the coupling configuration matrix representing topological structure of the network at time t, in which $C_{ij}(t)$ is defined as follows: If there is a connection from node i to node j(i ≠ j),then $C_{ij} = C_{ji} = 1$, otherwise $C_{ij} = C_{ji} = 0$ and the diagonal elements of matrix C(t) are defined by:

$$C_{ii}(t) = -\sum_{j=1, j\neq i}^{N} C_{ij}(t), i = 1, 2, \ldots N \tag{6}$$

Lemma 4: If C satisfies the above conditions, then there exists a unitary matrix, $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)$, such that:

$$C^T\phi_k = \lambda_k\phi_k, k = 1, 2, \ldots, N \tag{7}$$

where, λ_i, i = 1, 2,…, N, are the eigenvalues of C.

Hereafter, the delayed dynamical network 5 is said to achieve (asymptotical) synchronization if:

$$x_1(t) = x_2(t) = \ldots = x_N(t) \quad t \to \infty \tag{8}$$

where, $s(t)\epsilon R^n$ is a solution of an isolate node, namely:

$$\dot{s}(t) = f(s(t)) \tag{9}$$

Throughout this study, we assume that s(t) is an orbital stable solution of the above system. Clearly, the stability of the synchronized states 8 of network 5 is determined by the dynamics of the isolate node, the coupling strength c, the inner-coupling matrix G, the outer-coupling matrix C and the time-delay constant τ.

Theorem 1: Consider the delayed dynamical network 8. Let:

$$0 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \tag{10}$$

be the eigenvalues of the outer-coupling matrix C. If the following N-1 of n-dimensional time-varying delayed differential equations are asymptotically stable or stable about their zero solutions:

$$\dot{w} = J(t)w(t) + c\lambda_i G(t)w(t-\tau) \tag{11}$$

where, $J(t) = f'(s(t)) \in R^{n\times n}$ is the Jacobian of f (x(t)) at s(t), then the synchronized states 8 are asymptotically stable or stable for the dynamical network 5.

Proof: Without loss of generality, let $x_1(t) = s(t)$ be the reference direction of the synchronous manifold $x_1(t) = x_2(t) = \ldots = x_N(t)$. Then we have:

$$e_1(t) = x_1(t) - s(t) \equiv 0 \tag{12}$$

and:

$$x_i(t) = s(t) + e_i(t), i = 1, 2, \ldots, N \tag{13}$$

Substituting 13 into network 5 yields:

$$\begin{aligned}\dot{e}_i(t) &= f(s(t)+e_i(t)) - f(s(t)) \\ &+ c\sum_{j=2}^{N} C_{ij} G(t) e_i(t-\tau), i = 2, 3, \ldots, N\end{aligned} \tag{14}$$

Denote:

$$\overline{e}(t) = (e_2(t), e_3(t), \ldots, e_N(t))^T$$

$$e_i(t) = (e_{i1}(t), e_{i2}(t), \ldots, e_{in})^T$$

$$\overline{S}(t) = (s(t), s(t), \ldots, s(t))^T \in R^{n(N-1)}$$

Then Eq. 14 can be written as:

$$\dot{\overline{e}}(t) = F(t, \overline{e}(t)) \tag{15}$$

For the dynamical network 5, to investigate the asymptotical stability or stability of its synchronous state, we only need to analyze the asymptotical stability or stability of the zero transverse errors of the synchronous manifold.

From Eq. 15, we know that its corresponding linear system at $\overline{e}(t) = 0$ is:

$$\dot{\overline{e}}(t) = DF(t, 0)\overline{e}(t) \tag{16}$$

Since, the Jacobian DF(t,x) is bounded and Lipschitz on Ω, uniformly in t, according to the Lyapunov converse theorem [21], the origin is an asymptotical stable or stable equilibrium point for the nonlinear system 15 if and only if it is an asymptotical stable or stable equilibrium point for the linear time-varying system 15.

According to Eq. 12-13 and 16, we get:

$$\begin{aligned}\dot{e}_i(t) &= Df(s(t))e_i(t) + c\sum_{j=1}^{N} C_{ij}G(t)e_j(t-\tau)\\ &= Df(s(t))e_i(t) + cG(t)(e_1(t-\tau), e_2(t-\tau), \ldots,\\ &\quad e_N(t-\tau))(C_{i1}, C_{i2}, \ldots, C_{iN})^T, i = 1, 2, \ldots, N\end{aligned} \tag{17}$$

where, $e(t) = (e_1(t), e_2(t), \ldots, e_N(t)) \in R^{n\times N}$, $Df(s(t)) \in R^{n\times n}$ is the Jacobian of f(x) at x = s(t). That is:

$$\dot{e}(t) = Df(s(t))e(t) + cG(t)e(t-\tau)C^T \tag{18}$$

Since, $e_1(t) \equiv 0, \overline{e}(t) \to 0$ as $t \to 0$ is equivalent to $e(t) \to 0$ as $t \to 0$.

From the hypothesis of Lemma 5, we have:

$$C^T\Phi = \Phi\Lambda \tag{19}$$

where, $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. According to Lemmas 5, $\lambda_1 = 0$ for all $t > t_0$ and $\Phi^{-1} = (\phi'_1, \phi'_2, \ldots, \phi'_N)^T$ with $\phi'_1 = (1, 1, \ldots, 1)$ for all $t > t_0$.

Consider the following nonsingular linear transformation:

$$e(t) = v(t)^{-1} \tag{20}$$

According to Eq. 18, the matrix vector $v(t) = (v_1(t), v_2(t), \ldots, v_N(t)) \in R^{n\times N}$ satisfies the equation:

$$\dot{v}(t) = Df(s(t)v(t) + cG(t)v(t-\tau))\Lambda \tag{21}$$

namely:

$$\dot{v}_i(t) = Df(s(t)v(t) + c\lambda_i G(t)v_i(t-\tau))\Lambda \tag{22}$$

Thus, we have transformed the stability problem of the synchronized states 8 to the stability problem of the N pieces of n-dimensional linear time-varying delayed differential Eq. 22. Note that $\lambda_1 = 0$ corresponding to the synchronization of the system states 8, where the state s(t) is an orbital stable solution of the isolate node as assumed above in Eq. 8. If the following N-1 pieces of n-dimensional linear time-varying delayed differential equations:

$$\dot{v}_i(t) = Df(s(t)v_i(t) + c\lambda_i G(t)v_i(t-\tau))\Lambda \tag{23}$$

are asymptotically stable or stable, then e(t) will tend to the origin asymptotically which implies the synchronized states 8 are asymptotically stable or stable.

The proof is thus completed.

Then we will formulate main results based on the above preliminaries. In terms of Lemmas 1, 2 and 4, we can get the following stability criteria of synchronization for complex networks with time-varying inner-coupling function and time delays.

Theorem 2: Let J(t) be a constant matrix. The synchronized states 8 of network 5 are stable if the following conditions are satisfied:

$$\dot{x} = J(t)x \tag{24}$$

- Zero solution of system 24 is stable
- $\int_{t_0}^{\infty} \|c\lambda_i G(t)\| d_t < \infty$

Proof: If zero solution of system 24 is stable and:

$$\int_{t_0}^{\infty} \|c\lambda_i G(t)\| d_t < \infty$$

then the linear time-delay system 11 is stable according to Lemma 1.Then we resort to Lemma 4 and the synchronized states 8 of network 5 are stable when the linear time-delay system 11 is stable.

Theorem 3: Let J(t) be a continuous periodic function matrix. The synchronized states 8 of network 5 are asymptotically stable if the following conditions are satisfied:

$$\dot{x} = J(t)x,\ J(t+\omega) = J(t) \tag{25}$$

- Zero solution of system 25 is asymptotically stable
- $\int_{t_0}^{\infty} \|c\lambda_i G(t)\| d_t < \infty$

Proof: First of all, If zero solution of system 28 is asymptotically stable and:

$$\int_{t_0}^{\infty} \|c\lambda_i G(t)\| d_t < \infty$$

then the linear time-delay system 11 is asymptotically stable according to Lemma 2. Then we resort to Theorem 1 and the synchronized states 8 of network 5 are asymptotically stable when the linear time-delay system 11 is asymptotically stable.

TWO NUMERICAL EXAMPLES

In order to illustrate the main results of the above theoretical analysis, we first consider a lower-dimensional network model with five nodes, in which each node is a simple three-dimensional stable linear system described as follows (Li and Chen, 2004): $\dot{x}_1 = x_1$; $\dot{x}_2 = -2x_2$; $\dot{x}_3 = -3x_3$ which is asymptotically stable at s(t) = 0 and its Jacobian is J(t) = diag{-1 -2 -3}. Assume that the time-varying inner-coupling matrix is $G(t) = \mathrm{diag}\{e^{-t}, e^{-2t}, e^{-3t}\}$ and the outer-coupling matrix is C = [-2 1 0 0 1;1 -3 1 1 0;0 1 -2 1 0;0 1 1 -3 1;1 0 0 1 -2].

Obviously, G is an irreducible symmetric matrix satisfying condition 6. The eigenvalues of G are λ_i = 0, -1.382, -2.382, -3.618, -4.618. For clearer visions, we take the coupling strength c = 0.2 and time delay $\tau = 2$.

In terms of Theorem 2, if the condition 1 and 2 are satisfied, then it is inferred that for any delay the synchronization of the complex network can be achieved. First of all, we begin to verify whether the condition 1 is satisfied. Clearly, J(t) is a constant matrix. It is easy to verify that the following inequality is established for all nonzero three-dimensional vectors.

Let y = (y_1 y_2 y_3) be a nonzero three-dimensional vector, then:

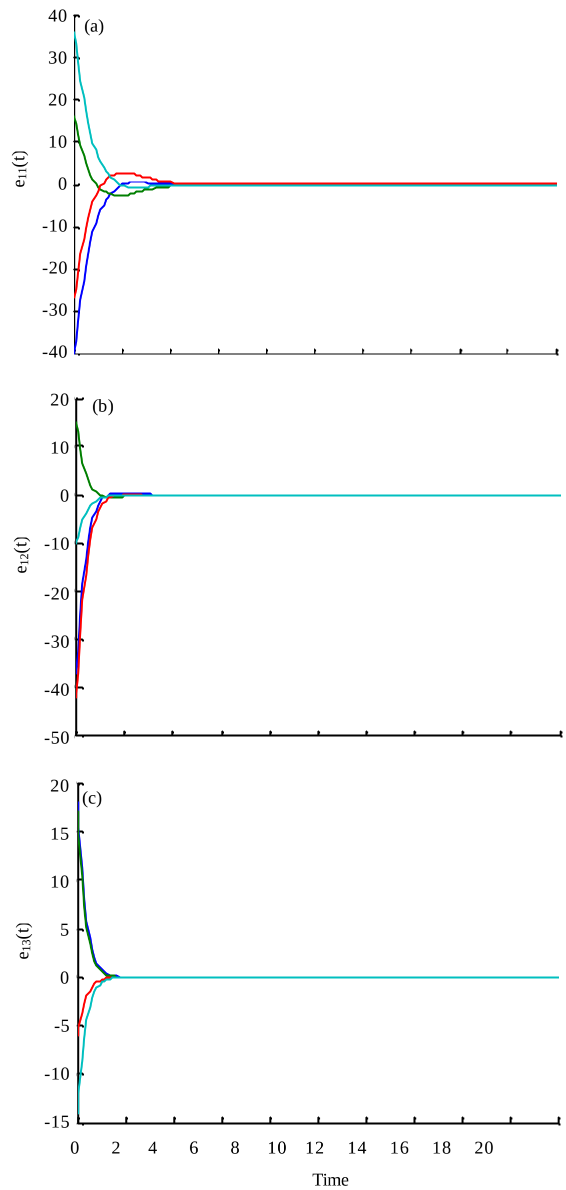

Fig. 1(a-c): Synchronization errors for the delayed network with c = 0.2 and $\tau = 2$

$$y\frac{J+J^T}{2}y^T = -\frac{1}{2}y_1^2 - y_2^2 - \frac{3}{2}y_3^2 < 0$$

From Lemma 3, we know that zero solution of system 24 is stable. Secondly, we will make sure whether the condition 2 of Theorem 2 is satisfied. Therefore, the synchronized states 8 of network 5 are stable. In Fig. 1, we plot the curves of the synchronization errors between the states of node i and node i+1(that is, $e_{ij}(t) = x_{ij}(t)-x_{i+1,j}(t)$), for i = 1, 2, 3, 4, j = 1, 2, 3, with the coupling strength c = 0.2 and time delay $\tau = 2$.

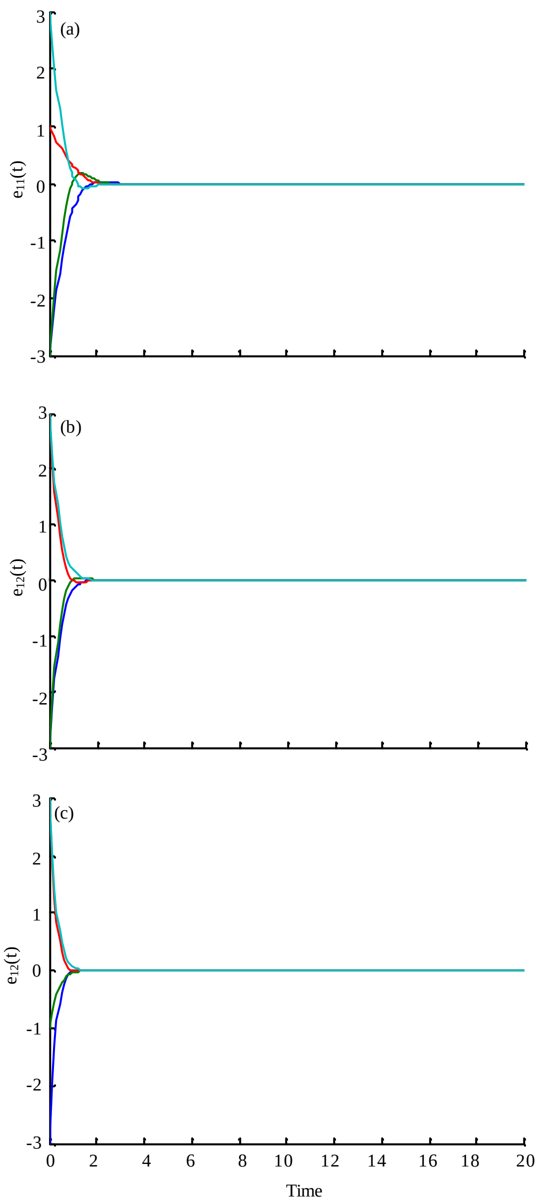

Fig. 2(a-c): Synchronization errors for the delayed network with c = 0.5 and $\tau = 1$

Then, we start to consider a lower-dimensional network model with five nodes, in which each node is a simple three-dimensional system as follows:

$$\dot{x}_1 = -(2-\cos(t))x_1$$

$$\dot{x}_2 = -(3-\cos(t))x_2$$

$$\dot{x}_3 = -(4-\cos(t))x_3$$

and its Jacobian is J(t) = diag{-2+cost -3+cost -4+cost}. Here, the time-varying inner-coupling matrix and the outer-coupling matrix are the same as the above example. Then, we will make sure whether the conditions of Theorem 3 are satisfied.

Let y = (y_1, y_2, y_3) be a nonzero three-dimensional vector, then:

$$y\frac{J+J^T}{2}y^T = (-2+\cos(t))y_1^2$$
$$+(-3+\cos(t))y_2^2 + (-4+\cos(t))y_3^2 < 0$$

Obviously, J(t) is a continuous periodic function matrix. From Lemma 3, we know that zero solution of system 25 is asymptotically stable. Secondly, we will make sure whether the condition 2 of Theorem 2 is satisfied. The rest part is similar to the above example. Therefore, the synchronized states 8 of network 5 are asymptotically stable. In Fig. 2, we also plot the curves of the synchronization errors between the states of node i and node i+1 with the coupling strength c = 0.5 and time delay $\tau = 1$.

CONCLUSION

This study mainly focused on the synchronization of complex dynamical networks with time-varying inner-coupling functions and time delays. According to the stability theory of the linear time-delay system, we have obtained new general stability criteria. By means of these criteria, we have avoided constructing the Lyapunov function, to investigate the stability of synchronization state. And two examples are numerically investigated.

ACKNOWLEDGMENTS

This work was supported by the Natural Science Foundation of China under Grant No. 60872073, the Natural Science Foundation of Fujian Province (Grant No. 2013H2002) and the Foundation of Huaqiao University under Grant No. 12BS228 and No. 13BS103.

REFERENCES

Albert, R. and A.L. Barabasi, 2002. Statistical mechanics of complex networks. Rev. Modern Phys., 74: 47-97.

Barabasi, A.L. and R. Albert, 1999. Emergence of scaling in random networks. Science, 286: 509-512.

Erdos, P. and A. Renyi, 1960. On the evolution of random graphs. Publ. Math. Inst. Hungarian Acad. Sci., 5: 17-61.

Hirsch, M.W., S. Smale and R.L. Devaney, 2008. Differential Equations, Dynamical Systems and an Introduction to Chaos. 2nd Edn., Posts and Telecom Press, Beijing, China.

Hoang, T.M., 2011. Recent progress in synchronization of multiple time delay systems. Chaos Solitons Fractals, 44: 48-57.

Li, C.G. and G. Chen, 2004. Synchronization in general complex dynamical networks with coupling delays. Physica A, 343: 263-278.

Peng, H.P., N. Wei, L.X. Li, W.S. Xie and Y.X. Yang, 2010. Models and synchronization of time-delayed complex dynamical networks with multi-links based on adaptive control. Phys.Lett. A, 374: 2335-2339.

Strogatz, S.H., 2001. Exploring complex networks. Nature, 410: 268-276.

Vidyasagar, M., 1993. Nonlinear Systems Analysis. 2nd Edn., Prentice Hall, Englewood Cliffs, NJ., USA.

Watts, D.J. and S.H. Strogatz, 1998. Collective dynamics of 'small-world' networks. Nature, 393: 440-442.