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Problems to Reselect Optimization to Pareto efficient solution for Linear Multi-objective Decision-making

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Abstract: Based on the solution relation between Linear Multi-Objective Programming (LMOP) and its corresponding weighted linear single objective programming, the dominant set of Pareto efficient solution of (LMOP) is defined. It is pointed out that $\bar{x}$ is an efficient solution of (LMOP) if and only if its dominant set is nonempty. It is proved that the efficient solutions of non-vertexes are considered to be eliminated relative to its some non-inferior vertexes. Further, it is seen that the linear multi-objective programming decision making is equivalent to a simple linear multi-objective programming which is easily transformed into a linear programming to get its solutions. Importantly, for (LMOP), some properties of non-inferior vertexes dominant set are discussed. By the definition of Pareto efficiency of feasible solution of (LMOP), it is proved that if $\bar{x}$ is not the efficient solution of (LMOP) or is the efficient solution of non-vertex of (LMOP), its Pareto efficiency $\eta = 0$; if $\bar{x}$ is non-inferior vertex of (LMOP), its Pareto efficiency $0<\eta\leq 1$. And the sum of the Pareto efficiency of all non-inferior vertexes is equal to one. It is important for us to order these non-inferior vertexes according to their Pareto efficiency.

Key words: Pareto efficient solution, dominant set, Convex polytope, non-inferior vertex, pareto efficiency

INTRODUCTION

As an important part of group decision making, Multi-Objective Decision Making (MODM) problems are wide spread in real decision situations. In general, the MODM problem can be dealt with by constructing a group utility function which often converts the MODM problem into a multi-objective programming (LMOP) problem (Wendell, 1980; Hwang and Lin, 1987). For MODM problems, if its objective function and constraint functions are linear, it will be called a linear multi-objective decision making. It is well known that the standard form of linear multi-objective decision making can be written as follows:

$$\text{(LMOP)}\ \min\ Z = Cx\ \text{s.t}\ \begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

where, $C = (c_{ij})_{p\times n}$, $A = (a_{ij})_{m\times n}$, $x = (x_1, x_2,\ldots, x_n)^T$, $b = (b_1, b_2,\ldots, b_m)^T$. Suppose that $m\leq n$, rank of the matrix of A is m and feasible solution set $S = \{x|Ax = b, x\geq 0\}$ is non-empty and bounded. it is noted that for Ax = b, if there are $a_{i0j}\geq 0$ (j =1, 2, ..., n) of the i_0 ($1 \leq i_0 \leq n$), $b_{i_0}>0$, $x_j \geq 0$ (j =1, 2, ..., n), the feasible set S is bounded which will be applied to the following example. Until recently, there exist some methods for solving the linear multi-objective programming. But the most classical method is the simplex algorithm of Zeleny (Zeleny, 1974; Yu and Zeleny, 1975; Yu, 1985; Benayoun *et al*., 1971) which belongs to a direct algorithm and can obtain all efficient solutions of the linear multi-objective programming. However, for a large linear multi-objective programming, there often exist a lot of solutions and it is difficult for the simplex algorithm to obtain these solutions which makes it not directly apply to management decision making practice. However, some new methods for the linear multi-objective programming are proposed from a particular aspect (Lin, 1976; Wang *et al*., 2005). Since, these new methods loss some information in the process of solution, these solution of the linear multi-objective programming will have no its popular signification. In this study, based on the Dominant set of feasible solutions of linear multi-objective programming, the linear multi-objective programming can be transformed into an equivalent simple multi-objective programming problem with limited feasible set. Further it can be transformed into some linear programming to get its solutions. Most importantly, Pareto efficiency of efficient solutions in linear multi-objective programming is defined to deal with the order of non-inferior vertexes.

The study is organized as follows. In Section 2, convex polytope is introduced and some of its properties

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are given. In Section 3, dominant set of feasible solutions of linear multi-objective programming is defined. Some of its properties are discussed. In Section 4, an equivalence theorem of linear multi-objective programming is proposed. In Section 5, some properties of non-inferior vertexes dominant set of the linear multi-objective programming and its Pareto efficient solutions are investigated. Conclusions are made in Section 6.

SOME PROPERTIES OF CONVEX POLYTOPE $S = \{x \mid Ax = b, x \geq 0\}$

Definition 1: $S = \{x \mid Ax = b, x \geq 0\}$ is called convex polytope if it is the feasible solution of Linear Multi-Objective Programming (LMOP).

Definition 2: Let C be a convex set of E^n. x is called a vertex of C if $x = \lambda x^{(1)}+(1 - \lambda x^{(2)})$ for any $x^{(1)}, x^{(2)} \in C$, $x^{(1)} \neq x^{(2)}$ and $\lambda \in (0, 1)$.

Lemma 1: Kou (1994) x is a vertex of convex polytope S if and only if A = [B, N], where B is nonsingular matrix of $m\times m$ and:

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0$$

It is noted that the feasible set S of Linear Multi-Objective Programming (LMOP) is a nonempty and bounded set. Let r be vertex numbers of convex polytope S. We have the following lemma.

Lemma 2: Kou (1994). The vertex numbers of convex polytope S is one at least and does not exceed C^m_n that is, $1 \leq r \leq C^m_n$.

It is easy to obtain all vertexes of convex polytope S by computer. In the following, the class of all non-inferior vertex of convex polytope S will be denoted $S^* = \{x^{(1)}, x^{(2)}, ..., x^{(r)}\}$, where $x^{(i)}$ (i = 1, 2, ..., r) is the ith non-inferior vertex of convex polytope S.

Lemma 3: Gu *et al*. (2005). For convex polytope S, any $x \in S$ can be expressed a convex combination by the vertex of S that is, for any $x \in S$:

$$x = \sum_{i=1}^{r} \lambda_i x^{(i)}$$

where, $0 \leq \lambda \leq 1$ (i = 1, 2, ..., r):

$$\sum_{i=1}^{r} \lambda_i = 1$$

and $x^{(i)}$ (i = 1, 2, ..., r) is the ith non-inferior vertex of convex polytope S.

DOMINANT SET OF FEASIBLE SOLUTIONS OF LINEAR MULTI-OBJECTIVE PROGRAMMING

For Linear Multi-Objective Programming (LMOP), its corresponding weighted linear single objective programming can be obtained as follows:

$$(LSOP)_{\bar{\lambda}} \ \min_{x \in S} Z_{\bar{\lambda}} = \bar{\lambda}^T Cx$$

Where:

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_p)^T \in E^p_{++} = \left\{\bar{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_p) \middle| \lambda_j > 0, j = 1, 2, \cdots, p\right\}$$

As for (LMOP) and $(LSOP)_{\bar{\lambda}}$, there are the following lemmas.

Lemma 4: For any $\bar{\lambda} \in E^p_{++}$, $(LSOP)_{\bar{\lambda}}$ has the optimal solution $\bar{x}$ and $\bar{x}$ is an efficient solution of (LMOP).

Lemma 5: For Linear Multi-Objective Programming (LMOP), let $\bar{x}$ be its any efficient solution, there must exist $\bar{\lambda} \in E^p_{++}$ such that $\bar{x}$ is the optimal solution of $(LSOP)_{\bar{\lambda}}$.

Lemma 6: For any $\bar{\lambda} \in E^p_{++}$, there exists $x^0 \in S^*$ such that x^0 is the optimal solution of the $(LSOP)_{\bar{\lambda}}$, sequentially, x^0 is an efficient solution of (LMOP) which is called a non-inferior vertex of (LMOP).

From the lemma 4 and 5, it is seen that that $\bar{x}$ is an efficient solution of (LMOP) is equivalent to what there exist $\bar{\lambda} \in E^p_{++}$ such that $\bar{x}$ is the optimal solution of $(LSOP)_{\bar{\lambda}}$. Let $\bar{x}$ be the optimal solution of $(LSOP)_{\bar{\lambda}}$ and $\bar{E} = \{\bar{\lambda} \mid \bar{\lambda} \in E^p_{++}\}$, the weight set $\bar{E}$ express the attribute character of $\bar{x}$, an efficient solution of (LMOP).

Definition 3: is called a dominant set of $\bar{x}$ if $\bar{x}$ is an efficient solution of (LMOP).

Theorem 1: $\bar{x}$ is an efficient solution of (LMOP) if and only if the dominant set $\bar{E}$ of $\bar{x}$ is nonempty.

In the following, the dominant set of $\bar{x}$ will be defined an empty set, i.e., $\bar{E} = \varphi$ when $\bar{x}$ is not an efficient solution of (LMOP).

Theorem 2: Let $\bar{x}$ be an efficient solution of (LMOP), then the dominant set $\bar{E}$ of $\bar{x}$ $\bar{E} = \left\{\bar{\lambda} \middle| \bar{\lambda} \in E^p_{++}, \bar{\lambda}^T C\bar{x} \leq \bar{\lambda}^T Cx^{(i)}\right\}$ is all vertexes of convex polytope S, where $x^{(i)} \in S^*$ (i = 1, 2, ..., r).

Proof: According to definition 3, we obtain $\bar{E} = \{\bar{\lambda} \mid \bar{\lambda} \in E^p_{++}, \ \bar{\lambda}^T C\bar{x} \leq \bar{\lambda}^T Cx, \forall x \in S\}$.

Let $\underline{E}=\bigcap_i\{\bar{\lambda}|\bar{\lambda}\in E^p_{++}, m\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^{(i)}\}$, it is obvious that $\bar{E}\subseteq\underline{E}$. In the following $\bar{E}\supseteq\underline{E}$ will be proofed.

For $\bar{\lambda}\in E^p_{++}$, suppose there exist $\bar{\lambda}\in\underline{E}$ and $?\notin\bar{E}$.

$\bar{\lambda}\notin\bar{E}$ implies that there exist x*∈S such that $\bar{\lambda}^T C\bar{x}>\bar{\lambda}^T Cx^*$. $?\in\underline{E}$ implies that $\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^{(i)}$.

So, for any $x^{(i)}\in S^*$, i = 1, 2,..., r, there exist $?\in E^p_{++}$ such that $\bar{\lambda}^T Cx^{(i)}>\bar{\lambda}^T Cx^*$, which is in contradiction with Lemma 6.

Thus $\bar{E}\supseteq\underline{E}$, i,e., $\bar{E}=\{\bar{\lambda}|\bar{\lambda}\in E^p_{++},\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^{(i)}\}$ = $\underline{E}=\bigcap_i\{\bar{\lambda}|\ \bar{\lambda}\in E^p_{++},\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^{(i)}\}$.

These complete the proof of Theorem 3.

Definition 4: Let x_1, x_2 be two different efficient solution of (LMOP) and E_1, E_2 are the dominant sets of x_1, x_2, respectively. x_1 is called to be eliminated relative to x_2 if $E_1\subseteq E_2$.

Remark 1 Definition 4 shows that if x_1 is excelledly included by x_2, in general, x_1 will be considered to be eliminated which reduces the numbers of efficient solution.

Theorem 3: Let $\bar{x}$ be efficient solution of non-vertex of (LMOP). $\bar{x}$ is considered to be eliminated relative to some non-inferior vertexes of (LMOP).

Proof: Let $\bar{x}$ be efficient solution of non-vertex of (LMOP). $\bar{E}$ is the dominant set of $\bar{x}$ and is not empty

According to Lemma 3, there exist $\bar{\mu}_1, \bar{\mu}_2,..., \bar{\mu}_r$, $0\le\bar{\mu}\le 1$, i = 1, 2,..., r:

$$\sum_{i=1}^{r}\bar{\mu}_i=1$$

such that:

$$\bar{x}=\sum_{i=1}^{r}\bar{\mu}_i x^{(i)}$$

that is, $\bar{x}$ can be denoted a convex combination of the vertex of S.

Suppose that $\bar{\mu}_1\ne 0$, 令:

$$\mu_1=\bar{\mu}_1,\ \mu_2=\sum_{i=2}^{r}\bar{\mu}_i$$

we have $\bar{x}=\mu_1 x^{(1)}+\mu_2 x^*$ where:

$$x^*=\sum_{i=2}^{r}\frac{\bar{\mu}_i x^{(i)}}{\mu_2}\in S \quad \text{且} \quad \mu_1+\mu_2=1$$

From Definition 3, for any $\bar{\lambda}\in\bar{E}$, x∈S, we have:

$$\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx \tag{1}$$

and $\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^{(1)}$; $\bar{\lambda}^T C\bar{x}\le\bar{\lambda}^T Cx^*$.

So, $\bar{\lambda}^T C(\mu_1 x^{(1)}+\mu_2 x^*)\le\bar{\lambda}^T C(\mu_1 x^{(1)}+\mu_2 x^{(1)})$, i.e., $\bar{\lambda}^T Cx^*\le\bar{\lambda}^T$ i.e., $\bar{\lambda}^T Cx^{(1)}\le\bar{\lambda}^T Cx^*$.

Thus for $\bar{\lambda}\in\bar{E}$, we have $\bar{\lambda}^T Cx^{(1)}=\bar{\lambda}^T Cx^*$ and:

$$\bar{\lambda}^T C\bar{x}=\bar{\lambda}^T C(\mu_1 x^{(1)}+\mu_2 x^*)=\bar{\lambda}^T Cx^{(1)} \tag{2}$$

Combing the inequality Eq. 1 with 2, we have that the dominant set of $x^{(1)}$ is not empty and $\bar{\lambda}$ belongs to the dominant set that is, $\bar{E}$ is a subset of this dominant set. Thus, $\bar{x}$ is considered to be eliminated relative to non-inferior vertex $x^{(1)}$ of (LMOP).

These complete the proof of Theorem 3.

Remark 2: From the Theorem 3, it is known that for linear multi-objective programming decision-making we only consider the non-inferior vertex.

EQUIVALENCE THEOREM OF LINEAR MULTI-OBJECTIVE PROGRAMMING AND ITS SIMPLE ALGORITHM

Definition 5: Let there be the same objective function between linear multi-objective programming (L_1) and (L_2) and S_1, S_2 be the sets of all non-eliminated efficient solutions of (L_1) and (L_2), respectively. We say that (L_1) is equivalent with (L_2) if S_1 = S_2 and for any $x\in S_1(S_2)$, the dominant set of x to (L_1) is equal to that of (L_2).

Remark 3: If linear multi-objective programming (L_1) is equivalent with (L_2), we will obtain all solutions of another linear multi-objective programming (L_1) by solving one linear multi-objective programming with less feasible set, by which original programming can be reduced.

As for simple linear multi-objective programming:

$$(\overline{LMOP})\ \min_{x\in S^*}\bar{Z}=Cx$$

where, S* is its feasible set.

From definition 3 and theorem 2, the following lemma is directly obtained.

Lemma 7: If $x_0\in S^*$, then $x_0\in S$, the dominant set of x_0 to $\overline{LMOP}$ is equal to that of (LMOP).

According to Lemma 7 and Theorem 3, it is easily to obtain the following lemma.

Lemma 8: x_0 is the efficient solution of $\overline{LMOP}$ if and only if it is non-inferior vertex of (LMOP).

According to Lemma 7 and Lemma 8, it is easily to obtain the following Equivalence theorem.

Theorem 3: Linear multi-objective programming is equivalent with the simple linear multi-objective programming.

Since, simple linear multi-objective programming is simpler than (LMOP), it is easy for to obtain its solutions by transforming $\overline{LMOP}$ into a linear programming.

For $x_{j_0} \in S^*$ and any $x_j \in S^*$, j = 1, ..., r. In the following x_{j_0} is simply denoted by x_0:

- If $Cx_j \geq 0$ and $Cx_j \neq 0$ (j = 1, ..., r), Let us consider the following linear programming:

$$(LP_1)\begin{cases} \min & \theta = \lambda^T Cx_0 \\ s.t & \lambda^T Cx_j \geq 1 \quad j = 1, \cdots, r \\ & \lambda \geq 0 \end{cases}$$

For the linear programming (LP_1), we have the following theorem.

Theorem 4: x_0 is the efficient solution of $\overline{LMOP}$ if there is $\lambda>0$, i.e., $\lambda \in E^p_{++}$, such that the optimal solution of (LP_1) $\theta = \lambda^T Cx_0 = 1$.

Proof: Firstly, we prove the sufficiency. If there is $\lambda>0$ such that the optimal solution of (LP_1) $\theta = \lambda^T Cx_0 = 1$, then for $\lambda>0$ and any $x_j \in S^*$, we have $\lambda^T Cx_0 \leq \lambda^T Cx_j$, that is, the dominant set of x_0 is not empty.

In the following, we shall prove the necessity.

If x_0 is the efficient solution of $\overline{LMOP}$, then its dominant set is not empty that is, there is $\lambda>0$ such that $\bar{\lambda}^T Cx_0 \leq \bar{\lambda}^T Cx_j$ for any $x_j \in S^*$.

Let $\mu = \bar{\lambda}^T Cx_0 > 0$ and:

$$\lambda = \frac{1}{\mu}\bar{\lambda}$$

we have there is $\lambda>0$ such that the optimal solution of (LP_1) $\theta = \lambda^T Cx_0 = 1$.

This completes the proof of Theorem 4:

- If there is $Cx_{j_k} \leq 0$ $(1 \leq j_k \leq r)$, there always is $y_0 \in E^p_{++}$ such that $Cx_j + y_0 \geq 0$ and $Cx_j + y_0 \neq 0$ (j = 1, ..., r). Let us consider the following linear programming:

$$(LP_2)\begin{cases} \min & \theta = \lambda^T C(x_0 + y_0) \\ s.t & \lambda^T C(x_j + y_0) \geq 1 \quad j = 1, \cdots, r \\ & \lambda \geq 0 \end{cases}$$

For the linear programming (LP_2), we have the following theorem.

Theorem 5: x_0 is the efficient solution of $\overline{LMOP}$ if there is $\lambda>0$, i.e., $\lambda \in y^p_{++}$, such that the optimal solution of (LP_2) $\theta = \lambda^T C(x_0+y_0) = 1$.

Proof: Since, the proof is similar to that of Theorem 4, we do not duplicate it and therefore here it is omitted.

SOME PROPERTIES OF NON-INFERIOR VERTEXES DOMINANT SET OF (LMOP) AND ITS PARETO EFFICIENCY

Let $\bar{S}^* = \{\bar{x}^{(1)}, \bar{x}^{(2)}, ..., \bar{x}^{(k)}\}$ where $\bar{x}^{(k)}$ is k-th non-inferior vertex of (LMOP), $0<k\leq r$. From Lemma 6 we have the following Lemma.

Lemma 9: For any $\bar{\lambda} \in E^p_{++}$, there is $x_0 \in \bar{S}^*$ such that x_0 is the optimal solution of $(LSOP)_{\bar{\lambda}}$, so x_0 is an efficient solution of (LMOP).

In the same way, From Theorem 2 we have the following Theorem.

Theorem 6: Let $\bar{x}$ be an efficient solution of (LMOP), then the dominant set $\bar{E}$ of $\bar{x}$ $\bar{E} = \bigcap\{\bar{\lambda} | \bar{\lambda} \in E^p_{++}, \bar{\lambda}^T C\bar{x} \leq \bar{\lambda}^T Cx^{(i)}\}$ is all non-inferior vertexes of (LMOP), where $x^{(i)} \in \bar{S}^*$ (i = 1, 2,..., k).

Theorem 7: Let $x^{(i)} \in \bar{S}^*$ (i = 1, 2, ..., k), the dominant set E_i has the following properties, respectively:

- (Convexity) For any $\lambda_1, \lambda_2 \in E_i$ and $\mu_1+\mu_2 = 1$, $0 \leq \mu_1, \mu_2 \leq 1$, implies $\mu_1\lambda_1+\lambda_2\mu_2 \in E_i$
- (Cone) For any $\lambda \in E_i$, $k > 0$, implies $k\lambda \in E_i$
- (Measurability) E_i (i = 1, 2, ..., k)is Legesgue measurable
- (Separation) E_i is separated from E_j, $i \neq j$, i, j = 1, 2,..., k and $m(E_i \cap E_j) = 0$, where m(*) denotes Legesgue measure of set "*"
- (Cover) $\bigcup_{i=1}^{k} E_i = E^p_{++}$

Proof: According to Definition 3, Theorem 1, Theorem 3 and Theorem 6, the properties 1, 2, 3 and 5 can be easily obtained. In the following we shall prove the property 4.

According to Theorem 6, we have:

$$E_i = \bigcap_l \{\lambda | \lambda \in E^p_{++}, \lambda^T Cx^{(i)} \leq \lambda^T Cx^{(l)}\}$$

where, $x^{(l)} \in \bar{S}$, l = 1,..., k, $l \neq i$:

$$\bigcap_j \{\lambda | \lambda \in E^p_{++}, \lambda^T Cx^{(j)} \le \lambda^T Cx^{(l)}\}$$

where, $x^{(l)} \in \bar{S}$, $l = 1, ..., k$, $l \ne j$. Thus:

$$E_i \cap E_j \subseteq \{\lambda | \lambda \in E^p_{++}, \lambda^T Cx^{(i)} = \lambda^T Cx^{(j)}\}$$

Since, $\lambda^T Cx^{(i)} - \lambda^T Cx^{(j)} = 0$ is hyperplane, we have $m(E_i \cap E_j) = 0$. For any $\lambda \in E_i$, $\lambda^T Cx^{(i)} - \lambda^T Cx^{(j)} \le 0$. For any $\lambda \in E_j$, $\lambda^T Cx^{(i)} - \lambda^T Cx^{(j)} \ge 0$. So, E_i is separated from E_j, $i \ne j$, $i, j = 1, 2, ..., k$. This completes the proof of property 4.

Definition 6:

$$\eta = m(\bar{E}) \big/ m(E^p_{++})$$

is called Pareto efficiency of $\bar{x}$ if $\bar{x}$ is feasible solution of linear multi-objective programming and $\bar{E}$ is the dominant set of it.

Remark 4: According to Theorem 1-2 and the properties 4, 5 of dominant set, it is seen that if $\bar{x}$ is not the efficient solution of (LMOP) or is the efficient solution of non-vertex of (LMOP), its Pareto efficiency $\eta = 0$; if $\bar{x}$ is non-inferior vertex of (LMOP), its Pareto efficiency $0 < \eta \le 1$.

From the properties 4-5 of dominant set, the following Theorem is directly obtained.

Theorem 8: Let $x^{(i)} \in \bar{S}^*$ ($i = 1, 2, ..., k$), η_i is the Pareto efficiency of $x^{(i)}$, then:

$$\sum_{i=1}^{k} \eta_i = 1$$

Remark 5: According to the above analysis, for linear multi-objective programming decision making problem, if the decision maker entirety does not know the importance of each objective of (LMOP), when he considers the weight of each objective, non-inferior vertex $x^{(i)}$ is the probability of what $x^{(i)}$ become a optimal solution. The Pareto efficiency η_i of non-inferior vertex $x^{(i)}$ is the probability of choosing $x^{(i)}$.

Theorem 9: Let $\bar{x}$ is the efficient solution of (LMOP), then $\bar{x}$ is the absolutely optimal solution of (LMOP) if and only if it's Pareto efficiency $\eta = 1$.

Proof: The necessity is obvious. We shall prove the sufficiency.

Suppose that $\bar{E}$ is the dominant set of $\bar{x}$, its Pareto efficiency $\eta = 1$, then we have $m(\bar{E}) = m(E^p_{++})$.

From the properties 1-2 of dominant set, we have that $\bar{E}^0$ is the interior of $\bar{E}$. So, $\bar{E}^0 = E^p_{++}$.

According to Definition 3, it is obvious that $\bar{E} = \bar{E}^0 = E^p_{++}$. For linear multi-objective programming, let:

$$C = (c_{ij})_{p \times n} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{pmatrix}$$

where, C_i ($i = 1, 2, ..., p$) is ith row vector of matrix C.

Suppose that $\bar{x}$ is not the absolutely optimal solution of (LMOP), then there is $x_0 \in S$ such that $C\bar{x} > Cx_0$, that is, there is at least a $i_0 (1 \le i_0 \le p)$ such that $C_{i_0}\bar{x} > C_{i_0}x_0$. Let:

$$\delta = \sum_{i \ne i_0} (C_i x_0 - C_i \bar{x})$$

if $n \to \infty$ we have:

$$C_{i_0}\bar{x} - \frac{\delta}{n} > C_{i_0}x_0$$

So, let:

$$\bar{\lambda} = (1, \underbrace{\frac{1}{n}, ..., \frac{1}{n}}_{p-1})^T \in E^p_{++}$$

we have $\bar{\lambda}^T C\bar{x} > \bar{\lambda}^T Cx_0$ which contradicts $\bar{\lambda} \in \bar{E}$. This completes the proof of Theorem.

Example: Let us consider the following linear multi-objective programming:

$$(L) \begin{cases} \min\ Z = (x_1 + 2x_2 + 4x_3,\ 2x_1 + x_2 + x_3)^T \\ 3x_1 - 5x_2 - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

Since, the rank of coefficient matrix $m = 2$, it is obvious that the feasible sets of linear multi-objective programming (L) is bounded. According to Lemma 1, it is easy to obtain two vertexes $(2, 1, 0)^T$ and $(1, 0, 2)^T$ of feasible set of (L), respectively.

According to Theorem 3 and 4, combing two vertexes $(2, 1, 0)^T$ and $(1, 0, 2)^T$ with linear programming (LP_1), it is seen that $(2, 1, 0)^T$ and $(1, 0, 2)^T$ are also non-inferior vertexes of (L) and are not eliminated.

For non-inferior vertex $(2, 1, 0)^T$ of (L), we have its dominant set:

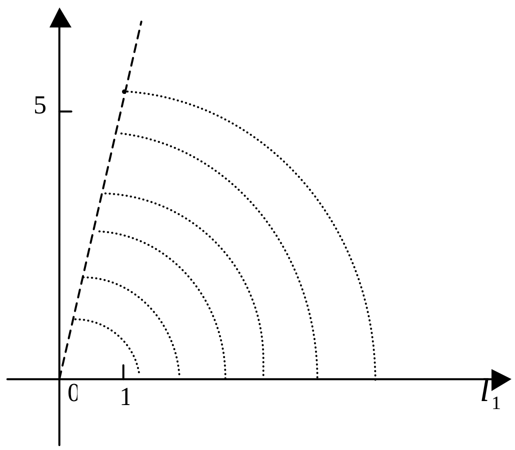

Chart1

$$\bar{E}_1 = \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in E^2_{++} \middle| 5\lambda_1 - \lambda_2 \ge 0 \right\}$$

which is expressed by chart 1.

From the chart 1, we can obtain the Pareto efficiency η_1 of non-inferior vertex $(2, 1, 0)^T$:

$$\eta_1 = \frac{m(\bar{E}_1)}{m(E^2_{++})} = \lim_{\rho \to +\infty} \frac{\int_0^{\text{arctg}5} \int_0^{\rho} r\,dr}{\int_0^{\frac{\pi}{2}} \int_0^{\rho} r\,dr} = \frac{2\text{arctg}5}{\pi} \approx 0.875$$

And then the Pareto efficiency η_2 of non-inferior vertex $(1, 0, 2)^T$:

$$\eta_2 = 1\text{-}0.875 = 0.125$$

So, for linear multi-objective programming (L), $(2, 1, 0)^T$ is the best Pareto solution.

CONCLUSION

In this study we draw some important conclusions as follows:

- Based on the definition of dominant set of efficient solution of (LMOP), it is proved that efficient solutions of non-vertex are considered to be eliminated relative to its some non-inferior vertex. In general, we only consider non-inferior vertex for linear multi-objective programming decision making
- It is proved that linear multi-objective programming (LMOP) decision making is equivalent to a simple linear multi-objective programming which is simpler than the (LMOP) decision making. It is easily transformed into a linear programming to get its solutions. Besides, all not eliminated solutions of (LMOP) can be obtained. The algorithm of the linear multi-objective programming decision making model belongs to a direct algorithm which does not loss any information
- For (LMOP), some properties of dominant set of non-inferior vertex are discussed. By the definition of Pareto efficiency of feasible solution, it is pointed that if $\bar{x}$ is not the efficient solution of (LMOP) or is the efficient solution of non-vertex of (LMOP), its Pareto efficiency $\eta = 0$; if $\bar{x}$ is non-inferior vertex of (LMOP), its Pareto efficiency $0<\eta\le 1$. And the sum of the Pareto efficiency of all non-inferior vertexes of (LMOP) is equal to one. It is important for us to order these non-inferior vertexes according to their Pareto efficiency

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