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Method for Multi-Attribute Group Decision Making Method Using Interval Type 2 Fuzzy Sets and Application to Teaching Quality Assessment

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Abstract: Type 2 fuzzy sets involve more uncertainties than type 1 fuzzy sets. Interval type 2 fuzzy set are quite useful to deal with fuzziness inherent in decision data and decision making process. For Multi-Attribute Group Decision Making (MAGDM) problems in which the attribute values and attribute weights are trapezoidal interval type 2 fuzzy sets (TIT2FSs), a new decision making method is proposed. A new ranking approach for TIT2FSs is given based on barycenter. The TIT2FSs weighted arithmetic average operator, TIT2FSs ordered weighted arithmetic average operator, TIT2FSs hybrid weighted arithmetic average operator and extended TIT2FSs hybrid weighted arithmetic average operator are developed. The individual comprehensive values of alternatives are derived through the extended TIT2FSs weighted arithmetic average operators. Combining the TIT2FSs hybrid weighted arithmetic average operator and the expert weights, the collective overall attribute values of alternatives are integrated. Then, the decision making results are given according to the collective comprehensive values of alternatives. Feasibility and effectiveness of the developed method are verified and demonstrated with a teaching quality assessment example.

Key words: Interval type 2 fuzzy set, multi-attribute group decision making, trapezoidal interval type 2 fuzzy set, aggregating operator

INTRODUCTION

Traditional type 1 fuzzy sets (Zadeh, 1965) have been widely used in the field of decision making. Many methods based on type 1 fuzzy sets have been presented for handling fuzzy Multiple Attributes Decision-making (MADM) problems and multiple attributes group decision-making (MAGDM) problems. Wang and Li (2013) proposed fuzzy LINMAP approach to heterogeneous MADM considering the comparisons of alternatives with hesitation degrees. Wan (2013a) developed power average operators of trapezoidal intuitionistic fuzzy numbers and application to MAGDM. Tan *et al.* (2010) presented a multi-criteria group decision making procedure using interval-valued intuitionistic fuzzy sets. Liu and Tan (2010) studied the interval MADM with preference information on attributes. Wan (2013b) proposed 2-tuple linguistic hybrid arithmetic aggregation operators and application to MAGDM. He and Deng (2011) presented an area-based approach to ranking fuzzy numbers in fuzzy decision making. Chen *et al.* (2013) investigated Services selection method based on multiple attribute decision making theory for group user. Sun and Sun (2012) developed induced fuzzy number intuitionistic fuzzy aggregating operators by means of Choquet integrals and their application in MAGDM. Yu *et al.* (2011) researched multi-criteria decision making based on Choquet integral under hesitant fuzzy environment.

The concept of a type 2 fuzzy set introduced by Zadeh (1975) is an extension of that of an ordinary fuzzy set (henceforth called type 1 fuzzy set). A type 2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership degree for each element of this set is a fuzzy set in (0, 1), unlike a type 1 fuzzy set where the membership degree is a real number in (0, 1) (Karnik and Mendel, 2001; Mendel *et al.*, 2006; Mendel and Liu, 2007; Wu and Mendel, 2007; Liu and Mendel, 2008). Though there have so many progress of applying type 1 fuzzy sets to the MADM and MAGDM problems, few authors research the application of type 2 fuzzy sets to the fields of MADM and MAGDM. To our best knowledge, there are a few of references investigated these topics up to now (Lee and Chen, 2008a, b; Chen and Lee, 2010a, b). Lee and Chen (2008a) proposed a new method for fuzzy MAGDM based on the arithmetic operations of interval type 2 fuzzy sets. Lee and Chen (2008b) extended the TOPSIS method to propose a new method for handling fuzzy MAGDM problems based on the ranking values of interval type 2 fuzzy sets. Chen and Lee (2010a) presented the arithmetic operations between interval type 2 fuzzy sets and the ranking values of interval type 2 fuzzy sets. Based on the proposed fuzzy ranking method and arithmetic operations, a new method to handle fuzzy MAGDM problems was proposed. Chen and Lee (2010b) presented an interval type 2 fuzzy TOPSIS method to handle fuzzy MAGDM problems based on interval type 2 fuzzy sets. However, the extended TOPSIS methods in Lee and Chen (2008b) and Chen and Lee (2010b) didn't consider the expert (or decision maker) weights. They only assumed that each expert has equal weight and used the simple arithmetic average to obtain the attribute weights. In practical MAGDM problems, the experts have their different cultural, educational backgrounds, experience and knowledge. Generally, different experts act as different roles in the process of MAGDM and thus have different importance degrees. Additionally, the strength of the trapezoidal membership functions defined by Chen and Lee (2010a) may be negative, which is not accordance with intuition, Thus, the ranking values of interval type 2 fuzzy sets is sometimes not reasonable.

Type 2 fuzzy sets allow us to handle the linguistic uncertainties as well as the numerical uncertainties. Type 2 fuzzy sets involve more uncertainties than type 1 fuzzy sets. A trapezoidal interval type 2 fuzzy set (TIT2FS) is a special kind of type 2 fuzzy sets. TIT2FS uses the reference points in the universe of discourse and the heights of the upper and the lower membership functions of interval type 2 fuzzy sets to characterize interval type 2 fuzzy sets. TIT2FSs can provide us with additional degrees of freedom to represent the uncertainty and fuzziness of the real world (Chen and Lee, 2010b). Moreover, TIT2FSs provide us with a useful way to handle fuzzy MAGDM problems in a more flexible and more intelligent manner due to the fact that it uses interval type 2 fuzzy sets rather than traditional type 1 fuzzy sets to represent the evaluating values and the weights of attributes. Therefore, the MAGDM problems based on TIT2FSs are of great importance for scientific researches and real applications.

However, to overcome the drawbacks of Lee and Chen (2008b) and Chen and Lee (2010a, b), there exist some difficulties and challenges, such as how to incorporate the information of expert weights into the aggregation operator, how to geometrically develop a new and reasonable ranking method of TIT2FSs. Consequently, the aim of this study is to study the new ranking method of TIT2FSs, develop some arithmetic aggregation operators for TIT2FSs and then propose a new MAGDM method based on type 2 fuzzy sets. Not only the proposed ranking method of TIT2FSs has intuitive and geometric meaning but also the developed TIT2FSs weighted arithmetic average operator and TIT2FSs hybrid weighted arithmetic average operator can sufficiently take the expert weights into consideration.

INTERVAL TYPE 2 FUZZY SETS

Definition and operation laws of interval type 2 fuzzy sets

Definition 1: Mendel *et al.* (2006) a type 2 fuzzy set A in the universe of discourse x can be represented by $A = \{(x,u)|\forall x\in X,\ \forall u\in J_X\subseteq[0,1],\ 0\le\mu_A(x,u)\le 1\}$, where J_X denotes an interval in (0,1), μ_A is a type 2 membership function. Moreover, the type 2 fuzzy set A also can be represented by $A = \int_{x\in X}\int_{u\in Jx}\mu_A(x,u)/(x,u)$, where $J_X\subseteq[0,1]$ and $\int\int$ denotes union over all admissible x and u.

Definition 2: Mendel *et al.* (2006) let A be a type 2 fuzzy set in the universe of discourse X represented by a type 2 membership function μ_A. If all $\mu_A(u,x) = 1$, then A is called an interval type 2 fuzzy set. An interval type 2 fuzzy set can be regarded as a special case of a type 2 fuzzy set, represented as follows: $A = \int_{x\in X}\int_{u\in Jx}1/(x,u)$, where $J_X\subseteq[0,1]$.

Definition 3: Mendel *et al.* (2006) the upper membership function and the lower membership function of an interval type 2 fuzzy set are the type 1 membership functions, respectively.

Definition 4: Lee and Chen (2008a) a trapezoidal interval type 2 fuzzy set (TIT2FS) A_i is defined as follows:

$$A_i = (A_i^U, A_i^L) = ((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(A_i^U), H_2(A_i^U)),\ (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(A_i^L), H_2(A_i^L)))$$

where A_i^U and A_i^L are type 1 fuzzy sets, a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L and a^L_{i4} are the reference points of the interval type 2 fuzzy set A_i, $H_j(A_i^U)$ denotes the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function A_i^U, $1\le j\le 2$, $H_j(A_i^U)$ denotes the membership value of the element $a_{i(j+1)}^L$ in the lower trapezoidal membership function A_i^L, $1\le j\le 2$, $H_1(A_i^U)\in[0,1]$, $H_2(A_i^U)\in[0,1]$, $H_1(A_i^L)\in[0,1]$, $H_2(A_i^L)\in[0,1]$ and $1\le i\le n$.

The upper membership function and the lower membership function of a TIT2FS are the type 1 membership functions, respectively. A TIT2FS A_i is depicted as in Fig. 1.

Example 1: There is a TIT2FS A = (1, 3, 7, 9; 0.8, 0.6), (2, 4, 6, 8; 0.2, 0.4). Then, when x = 5, its upper membership degree being a TIT2FS A_1 is 0.7, its lower membership degree being a TIT2FS A_1 is 0.3.

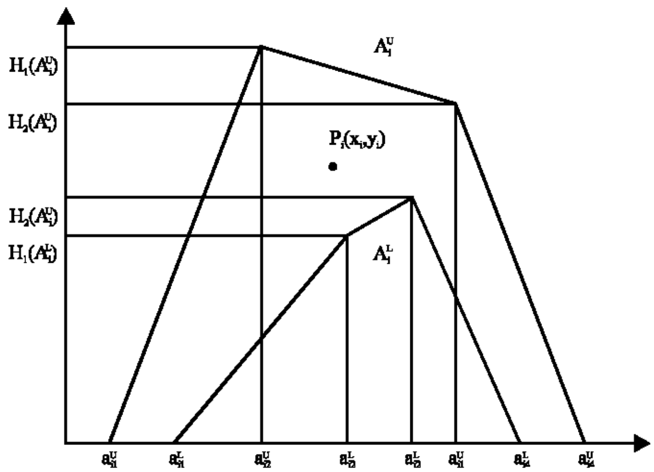

Fig. 1: Upper trapezoidal membership function membership function A_i^U and the lower trapezoidal membership function membership function A_i^L of a TIT2FS A_i

Definition 5: Lee and Chen (2008a) lLet A_1 and A_2 be two TIT2FSs, $A_i = (A_i^U, A_i^L) = ((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, H_1(A_i^U), H_2(A_i^U)), (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L, H_1(A_i^L), H_2(A_i^L)))$, i = 1, 2. K>0 is a crisp real number. The operation laws of TIT2FSs are defined as follows:

$$A_1 + A_2 = (A_1^U, A_1^L) + (A_2^U, A_2^L) = ((a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))), (a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))))$$

$$A_1 - A_2 = (A_1^U, A_1^L) - (A_2^U, A_2^L) = ((a_{11}^U - a_{24}^U, a_{12}^U - a_{23}^U, a_{13}^U - a_{22}^U, a_{14}^U - a_{21}^U; \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))), (a_{11}^L - a_{24}^L, a_{12}^L - a_{23}^L, a_{13}^L - a_{22}^L, a_{14}^L - a_{21}^L; \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))))$$

$$A_1 A_2 = (A_1^U, A_1^L)(A_2^U, A_2^L) = ((a_{11}^U a_{21}^U, a_{12}^U a_{22}^U, a_{13}^U a_{23}^U, a_{14}^U a_{24}^U; \min(H_1(A_1^U), H_1(A_2^U)), \min(H_2(A_1^U), H_2(A_2^U))), (a_{11}^L a_{21}^L, a_{12}^L a_{22}^L, a_{13}^L a_{23}^L, a_{14}^L a_{24}^L; \min(H_1(A_1^L), H_1(A_2^L)), \min(H_2(A_1^L), H_2(A_2^L))))$$

$$kA_i = ((ka_{i1}^U, ka_{i2}^U, ka_{i3}^U, ka_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (ka_{i1}^L, ka_{i2}^L, ka_{i3}^L, ka_{i4}^L; H_1(A_i^L), H_2(A_i^L))),$$

$$\frac{A_i}{k} = ((\frac{1}{k} a_{i1}^U, \frac{1}{k} a_{i2}^U, \frac{1}{k} a_{i3}^U, \frac{1}{k} a_{i4}^U; H_1(A_i^U), H_2(A_i^U)), (\frac{1}{k} a_{i1}^L, \frac{1}{k} a_{i2}^L, \frac{1}{k} a_{i3}^L, \frac{1}{k} a_{i4}^L; H_1(A_i^L), H_2(A_i^L)))$$

A ranking method for interval TIT2FS based on barycenter: Since the upper and lower membership functions of a TIT2FS are the type 1 membership functions. From Fig. 1, we can view the region between the upper and lower membership functions as sheets with uniform density and calculate the barycentric coordinate of the region.

Definition 6: For the image of upper and lower membership functions of a TIT2FS A_i, let the barycentric coordinates of the region between the upper and lower membership functions be $P_i(x_i, y_i)$. Then, the ranking value of A_i is defined as follows:

$$R(A_i) = x_i, y_i \tag{1}$$

Since a TIT2FS A_i is characterized by the upper and lower membership functions, the membership degree of element $x \epsilon A_i$ lies in the region between the upper and lower membership functions. The total mass of the sheet focuses on the barycenter, the horizontal coordinate x_i and vertical coordinate y_i of barycenter represent, respectively the value of the element x and its corresponding membership degree of $x \epsilon A_i$. Equation 1 defines the ranking value of A_i by the product of horizontal and vertical coordinates of barycenter. So the ranking value defined by Eq. 1 simultaneously reflects the element value and the corresponding membership degree of element belonging to A_i and its geometrical meaning is very clear.

Remark 1: By the calculous theory, the barycentric coordinates can be easily obtained. We take Fig. 1 as an example to illustrate the computing process.

The area of the sheet is:

$$M_i = \int_{a_{i1}^U}^{a_{i2}^U} \frac{x - a_{i1}^U}{a_{i2}^U - a_{i1}^U} H_1(A_i^U)dx + \int_{a_{i2}^U}^{a_{i3}^U} [H_1(A_i^U) - \frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U))]dx$$
$$+\int_{a_{i3}^U}^{a_{i4}^U} -\frac{x - a_{i4}^U}{a_{i4}^U - a_{i3}^U} H_2(A_i^U)dx - \int_{a_{i1}^L}^{a_{i2}^L} \frac{x - a_{i1}^L}{a_{i2}^L - a_{i1}^L} H_1(A_i^L)dx -$$
$$\int_{a_{i2}^L}^{a_{i3}^L} [H_1(A_i^L) - \frac{x - a_{i2}^L}{a_{i3}^L - a_{i2}^L}(H_1(A_i^L) - H_2(A_i^L))]dx - \int_{a_{i3}^L}^{a_{i4}^L} -\frac{x - a_{i4}^L}{a_{i4}^L - a_{i3}^L} H_2(A_i^L)dx$$

The static moment to x-axis is:

$$M_{ix} = \frac{1}{2}\{\int_{a_{i1}^U}^{a_{i1}^L} [\frac{H_1(A_i^U)}{a_{i2}^U - a_{i1}^U}(x - a_{i1}^U)]^2 dx + \int_{a_{i1}^L}^{a_{i2}^U} [\frac{H_1(A_i^U)}{a_{i2}^U - a_{i1}^U}$$
$$(x - a_{i1}^U)]^2 - [\frac{H_1(A_i^L)}{a_{i2}^L - a_{i1}^L}(x - a_{i1}^L)]^2 dx + \int_{a_{i2}^U}^{a_{i2}^L} [H_1(A_i^U) -$$
$$\frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U))]^2 - [\frac{H_1(A_i^L)}{a_{i2}^L - a_{i1}^L}(x - a_{i1}^L)]^2 dx$$
$$+\int_{a_{i2}^L}^{a_{i3}^L} [H_1(A_i^U) - \frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U))]^2 -$$
$$[H_1(A_i^L) - \frac{x - a_{i2}^L}{a_{i3}^L - a_{i2}^L}(H_1(A_i^L) - H_2(A_i^L))]^2 dx$$
$$+\int_{a_{i3}^L}^{a_{i3}^U} [H_1(A_i^U) - \frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U))]^2$$
$$-[\frac{x - a_{i4}^L}{a_{i4}^L - a_{i3}^L} H_2(A_i^L)]^2 dx + \int_{a_{i3}^U}^{a_{i4}^L} [\frac{x - a_{i4}^U}{a_{i4}^U - a_{i3}^U} H_2(A_i^U)]^2 -$$
$$[\frac{x - a_{i4}^L}{a_{i4}^L - a_{i3}^L} H_2(A_i^L)]^2 dx + \int_{a_{i4}^L}^{a_{i4}^U} [\frac{x - a_{i4}^U}{a_{i4}^U - a_{i3}^U} H_2(A_i^U)]^2 dx\}$$

The static moment to y-axis is:

$$M_{iy} = \int_{a_{i1}^U}^{a_{i1}^L} \frac{H_1(A_i^U)}{a_{i2}^U - a_{i1}^U}(x - a_{i1}^U)xdx + \int_{a_{i1}^L}^{a_{i2}^U} x[\frac{H_1(A_i^U)}{a_{i2}^U - a_{i1}^U}$$
$$(x - a_{i1}^U) - \frac{H_1(A_i^L)}{a_{i2}^L - a_{i1}^L}(x - a_{i1}^L)]dx + \int_{a_{i2}^U}^{a_{i2}^L} x[H_1(A_i^U) -$$
$$\frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U)) - \frac{H_1(A_i^L)}{a_{i2}^L - a_{i1}^L}(x - a_{i1}^L)]dx$$
$$+\int_{a_{i2}^L}^{a_{i3}^L} x[H_1(A_i^U) - \frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U)) -$$
$$H_1(A_i^L) + \frac{x - a_{i2}^L}{a_{i3}^L - a_{i2}^L}(H_1(A_i^L) - H_2(A_i^L))]dx + \int_{a_{i3}^L}^{a_{i3}^U} x[H_1(A_i^U)$$
$$-\frac{x - a_{i2}^U}{a_{i3}^U - a_{i2}^U}(H_1(A_i^U) - H_2(A_i^U)) - \frac{x - a_{i4}^L}{a_{i4}^L - a_{i3}^L} H_2(A_i^L)]dx$$
$$+\int_{a_{i3}^U}^{a_{i4}^L} x[\frac{x - a_{i4}^U}{a_{i4}^U - a_{i3}^U} H_2(A_i^U) - \frac{x - a_{i4}^L}{a_{i4}^L - a_{i3}^L} H_2(A_i^L)]dx + \int_{a_{i4}^L}^{a_{i4}^U}$$
$$x[\frac{x - a_{i4}^U}{a_{i4}^U - a_{i3}^U} H_2(A_i^U)]dx$$

Hence, the horizontal and vertical coordinates of barycenter $P_i(x_i, y_i)$ are respectively:

$$x_i = M_{iy}/M_i,\ y_i = M_{ix}/M_i$$

Remark 2: For two TIT2FSs $A_s = (A_s^U, A_s^L)$ and $A_t = (A_t^U, A_t^L)$, Chen and Lee (2010a) defined the strength E_{ts} of A_t^U over A_s^U and thereby gave the ranking value of TIT2FS. It is obviously that the strength E_{ts} may be negative, which is not accordance with intuition. In our intuition, the strength of one fuzzy set over another is usually non-negative. However, the ranking method of TIT2FS proposed in this study is very intuitive and has clear geometrical mean and thus more reasonable than that proposed in Chen and Lee (2010a).

According to the ranking value of Eq. 1, a new ranking method between two TIT2FSs A_1 and A_2 is given as follows:

- If $R(A_1)<R(A_2)$, then A_1 is smaller than A_2, denoted by $A_1<A_2$
- If $R(A_1)>R(A_2)$, then A_1 is bigger than A_2, denoted by $A_1>A_2$
- If $R(A_1) = R(A_2)$, then A_1 and A_2 represent the same information, denoted by $A_1 = A_2$

Example 2: Let A_1 and A_2 be two TIT2FSs as follows:

- $A_1 = ((1, 3, 7, 9; 0.8, 0.6), (2, 4, 6, 8; 0.2, 0.4))$
- $A_2 = ((1.5, 3, 6.5, 8.5; 0.7, 0.5), (2, 4.5, 6, 8; 0.5, 0.3))$

Using the above method, we can get the coordinates of barycenters for A_1 and A_2 as follows:

- $M_1 = 3.0$, $M_{1x} = 1.16$, $M_{1y} = 13.8$, $x_1 = 4.6$, $y_1 = 0.3867$
- $M_2 = 1.6$, $M_{2x} = 0.69$, $M_{2y} = 8.1958$, $x_2 = 5.1223$, $y_2 = 0.4313$

By Eq. 1, the ranking values are obtained as follows: $R(A_1) = 1.7787$, $R(A_2) = 2.2090$. Since $R(A_1)<R(A_2)$, the ranking order is $A_1<A_2$.

Some arithmetic aggregation operators for TIT2FSs

Definition 7: Assume that $A_i = (A_i^U, A_i^L)$ (i = 1, 2,..., n) is a collection of the TIT2FSs. Let $\phi: T^n \to T$. If :

$$\phi_w(A_1, A_2, \cdots, A_n) = \sum_{i=1}^{n} w_i A_i \tag{2}$$

where, T is the set composed of all TIT2FSs, $w = (w_1, w_2,..., w_n)^T$, w_i is the weight of A_i, $0 \le w_i \le 1$, (i = 1, 2,..., n):

$$\sum_{i=1}^{n} w_i = 1$$

then the function ϕ is called the TIT2FS weighted arithmetic average (TIT2FS-WA) operator of dimension n. If $w_i = \frac{1}{n}$ $(i = 1, 2, \cdots, n)$, then TIT2FS-WA is reduced to TIT2FS arithmetic average (TIT2FS-A) operator.

Theorem 1: Let $A_i = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$ be a collection of the TIT2FSs, then their aggregated value by using TIT2FS-WA operator is also a TIT2FS and:

$$\begin{aligned}\phi_w(A_1, A_2, \cdots, A_n) = ((&\sum_{i=1}^{n} w_i a_{i1}^U, \sum_{i=1}^{n} w_i a_{i2}^U, \sum_{i=1}^{n} w_i a_{i3}^U, \\ &\sum_{i=1}^{n} w_i a_{i4}^U; \wedge_i H_1(A_i^U), \wedge_i H_2(A_i^U)), (\sum_{i=1}^{n} w_i a_{i1}^L, \sum_{i=1}^{n} w_i a_{i2}^L, \\ &\sum_{i=1}^{n} w_i a_{i3}^L, \sum_{i=1}^{n} w_i a_{i4}^L; \wedge_i H_1(A_i^L), \wedge_i H_2(A_i^L)))\end{aligned} \tag{3}$$

where, symbol $\wedge_i$ is the min operator on I.

Proof: Combining definition 5, Theorem 1 can be easily proven by using mathematical induction on n.

The TIT2FS-WA operator has some desirable properties, such as idempotency, boundedness, commutativity, monotonicity and so on.

Proposition 1 (Idempotency): If all $A_i = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$ are equal, i.e., $A_i = A = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$, then $\phi_w(A_1, A_2,..., A_n) = A$.

Proposition 2 (Boundedness): Let $A^+ = \max\{A_i|i = 1, 2,..., n\}$ and $A^- = \min\{A\ A_i|i = 1, 2,..., n\}$. Then:

$$A^- \leq \phi_w(A_1, A_2,..., A_n) \leq A^+$$

Proposition 3 (Commutativity): Let $(\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n)$ is any permutation of $(A_1, A_2,..., A_n)$. Then:

$$\phi_w(A_1, A_2, \cdots, A_n) = \phi_w(\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n)$$

Proposition 4 (Monotonicity): Let $(A_1, A_2,..., A_n)$ and $(B_1, B_2,..., B_n)$be two collections of the TIT2FSs and $A_i \leq B_i$ $(i = 1, 2,..., n)$. Then:

$$\phi_w(A_1, A_2,..., A_n) \leq \phi_w(B_1, B_2,..., B_n)$$

Definition 8: Assume that $A_i = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$ is a collection of the TIT2FSs. Let $\varphi: T^n \rightarrow T$. If:

$$\varphi_w(A_1, A_2, \cdots, A_n) = \sum_{i=1}^{n} w_i A_{\sigma(i)} \tag{4}$$

where, $(w_1, w_2,..., w_n)^T$ is the weighted vector correlating with φ, $0 \leq w_i \leq 1$ $(i = 1, 2,..., n)$:

$$\sum_{i=1}^{n} w_i = 1,$$

$\sigma_{(1)},..., \sigma_{(n)}$ is a permutation of $(i = 1, 2,..., n)$ such that $A_{\sigma(i-1)} \geq A_{\sigma(i)}$ for any i. Then the function φ is called the TIT2FS ordered weighted average (TIT2FS-OWA) operator of dimension n.

Theorem 2: Let $A_i = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$ be a collection of the TIT2FSs, then their aggregated value by using TIT2FS-OWA operator is also a TIT2FS and:

$$\begin{aligned}\varphi_w(A_1, A_2, \cdots, A_n) = ((&\sum_{i=1}^{n} w_i a_{\sigma(i)1}^U, \sum_{i=1}^{n} w_i a_{\sigma(i)2}^U, \sum_{i=1}^{n} w_i a_{\sigma(i)3}^U \\ &, \sum_{i=1}^{n} w_i a_{\sigma(i)4}^U; \wedge_i H_1(A_{\sigma(i)}^U), \wedge_i H_2(A_{\sigma(i)}^U)), (\sum_{i=1}^{n} w_i a_{\sigma(i)1}^L, \\ &\sum_{i=1}^{n} w_i a_{\sigma(i)2}^L, \sum_{i=1}^{n} w_i a_{\sigma(i)3}^L, \sum_{i=1}^{n} w_i a_{\sigma(i)4}^L; \wedge_i H_1(A_{\sigma(i)}^L), \wedge_i H_2(A_{\sigma(i)}^L)))\end{aligned} \tag{5}$$

The TIT2FS-OWA operator also has some desirable properties, such as idempotency, boundedness, commutativity and so on.

It can be seen from definitions 7 and 8 that the TIT2FS-WA operator weights the TIT2FS arguments while the TIT2FS-OWA operator weights the ordered positions of the TIT2FS arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the TIT2FS-WA and TIT2FS-OWA operators. However, both the operators consider only one of them. To solve this drawback, based on definitions 2, 3 and 4, TIT2FS hybrid weighted average operator is developed in the following.

Definition 9: Let $A_i = (A_i^U, A_i^L)$ $(i = 1, 2,..., n)$ is a collection of the TIT2FSs. Let $\Phi: T^n \rightarrow T$. If:

$$\Phi_{w,\omega}(A_1, A_2, \cdots, A_n) = \sum_{i=1}^{n} w_i A'_{\sigma(i)} \tag{6}$$

where, $(w_1, w_2,..., w_n)^T$ is the weighted vector correlating with Φ, $0 \leq w_i \leq 1$ $(i = 1, 2,..., n)$:

$$\sum_{i=1}^{n} w_i = 1,\ A'_{\sigma(i)}$$

is the ith largest TIT2FS of TIT2FSs A'_i $(i = 1, 2,..., n)$ with $A'_i = n\omega_i A_i$, $\omega = (\omega_1, \omega_2,..., \omega_n)^T$, is the weighting vector of TIT2FSs A'_i $(i = 1, 2,..., n)$, $0 \leq \omega_i \leq 1$:

$$\sum_{i=1}^{n} \omega_i = 1$$

n is the balancing coefficient. Then the function Φ is called the TIT2FS hybrid weighted average (TIT2FS-HWA) operator of dimension n.

Proposition 5: The TIT2FS-OWA operator is a special case of the TIT2FS-HWA operator.

Proof: Let $\omega_i = 1/n$ (j = 1, 2,..., n), then $A'_i = n\omega_i A_i = A_i$ (i = 1, 2,..., n). This completes the proof of Theorem 2.

Proposition 6: The TIT2FS-WA operator is a special case of the TIT2FS-HWA operator.

Proof: Let $\omega_j = 1/n$ (j = 1, 2,..., n), then:

$$\Phi_{w,\omega}(A_1, A_2, \cdots, A_n) = \sum_{i=1}^{n} w_i A'_{\sigma(i)} = \sum_{i=1}^{n} \tfrac{1}{n} A'_{\sigma(i)}$$
$$= \sum_{i=1}^{n} \tfrac{1}{n} n\omega_i A_i = \sum_{i=1}^{n} \omega_i A_i = \phi_\omega(A_1, A_2, \cdots, A_n)$$

which completes the proof of Theorem 3.

From Proposition 5 and 6, we know that, TIT2FS-HWA operators first weights the given arguments and then reorders the weighted arguments in descending order and weights these ordered arguments and finally aggregates all the weighted arguments into a collective one. The TIT2FS-HWA operator generalizes both the TIT2FS-WA and TIT2FS-OWA operators and reflects the important degrees of both the given TIT2FSs and the ordered positions of the TIT2FSs.

Remark 3: In MAGDM, since there are multiple experts participating decision making together, different experts should allocated different weights. The TIT2FS-HWA operator can be used to integrate the individual comprehensive attribute values of alternatives into the collective ones by sufficiently considering the expert weights.

Definition 10: Let $A_i = (A_i^U, A_i^L)$ (i = 1, 2,..., n) is a collection of the TIT2FSs. Let Γ: $T^n \rightarrow T$. If:

$$\Gamma_w(A_1, A_2, \cdots, A_n) = \sum_{i=1}^{n} w_i A_i \quad (7)$$

where, w_i is the weight of A_i and w_i is also a TIT2FS $w_i = ((w_{i1}^U, w_{i2}^U, w_{i3}^U, w_{i4}^U; H_1(w_i^U), H_2(w_i^U)), (w_{i1}^L, w_{i2}^L, w_{i3}^L, w_{i4}^L; H_1(w_i^L), H_2(w_i^L)))$, then the function Γ is called the extended TIT2FS weighted average (ETIT2FS-WA) operator of dimension n.

Theorem 3: Let $A_i = (A_i^U, A_i^L)$ (i = 1, 2,..., n) be a collection of the TIT2FSs, then their aggregated value by using ETIT2FS-WA operator is also a TIT2FS and:

$$\begin{aligned}\Gamma_w(A_1, A_2, \cdots, A_n) = & ((\sum_{i=1}^{n} w_{i1}^U a_{i1}^U, \sum_{i=1}^{n} w_{i2}^U a_{i2}^U, \sum_{i=1}^{n} w_{i3}^U a_{i3}^U, \sum_{i=1}^{n} w_{i4}^U a_{i4}^U; \\ & \wedge_i \min\{H_1(A_i^U), H_1(w_i^U)\} \wedge_i \min\{H_2(A_i^U), H_2(w_i^U)\}), \\ & (\sum_{i=1}^{n} w_{i1}^L a_{i1}^L, \sum_{i=1}^{n} w_{i2}^L a_{i2}^L, \sum_{i=1}^{n} w_{i3}^L a_{i3}^L, \sum_{i=1}^{n} w_{i4}^L a_{i4}^L; \wedge_i \\ & \min\{H_1(A_i^L), H_1(w_i^L)\} \wedge_i \min\{H_2(A_i^L), H_2(w_i^L)\})) \end{aligned} \quad (8)$$

Remark 4: In MAGDM with TIT2FSs, since the attribute values are expressed with TIT2FSs, experts also maybe give the attribute weights in the form of TIT2FSs. Thus, the ETIT2FS-WA operator can be directly used to integrate the attribute values of alternatives into the individual comprehensive attribute values of alternatives.

MAGDM MODEL AND METHOD BASED ON TIT2FSS

Description of MAGDM problem based on TIT2FSs: For some fuzzy MAGDM problem, denote an alternative set by $A = \{A_1,..., A_m\}$, an attribute set by $F = \{a_1, a_2,.., a_n\}$ and an expert set by $D = \{d_1, d_2,..., d_t\}$. Let $\lambda = (\lambda_1, \lambda_2,..., \lambda_t)^T$ be the weight vector of experts:

$$0 \le \lambda_k \le 1, \sum_{k=1}^{t} \lambda_k = 1$$

Assume that the weight of attribute a_j given by expert d_k is a TIT2FS as follows:

$$W_j^k = (W_j^{kU}, W_j^{kL}) = ((w_{j1}^{kU}, w_{j2}^{kU}, w_{j3}^{kU}, w_{j4}^{kU}; H_1(W_j^{kU}), H_2(W_j^{kU})), (w_{j1}^{kL}, w_{j2}^{kL}, w_{j3}^{kL}, w_{j4}^{kL}; H_1(W_j^{kL}), H_2(W_j^{kL})))$$

Suppose that the fuzzy rating of an alternative A_i on an attribute a_j given by expert d_k is a TIT2FS represented as follows:

$$A_{ij}^k = (A_{ij}^{kU}, A_{ij}^{kL}) = ((a_{ij1}^{kU}, a_{ij2}^{kU}, a_{ij3}^{kU}, a_{ij4}^{kU}; H_1(A_{ij}^{kU}), H_2(A_{ij}^{kU})), (a_{ij1}^{kL}, a_{ij2}^{kL}, a_{ij3}^{kL}, a_{ij4}^{kL}; H_1(A_{ij}^{kL}), H_2(A_{ij}^{kL})))$$

where, A_{ij}^{kU} denotes the upper membership degree to which the alternative A_i belongs to TIT2FS A_{ij}^k on the attribute a_j, A_{ij}^{kL} denotes the lower membership degree to which alternative A_i belongs to TIT2FS A_{ij}^k on the attribute a_j and $H_1(A_{ij}^{kU}) \in [0,1]$. Hence, a fuzzy MAGDM problem based on TIT2FSs can be concisely expressed in TIT2FS matrix format $H_2(A_{ij}^{kU}) \in [0,1]$, $H_1(A_{ij}^{kL}) \in [0,1]$, $H_2(A_{ij}^{kL}) \in [0,1]$.

The MAGDM problem considered in this study is to determine the best alternative from the finite alternative set according to the matrix $\tilde{A}^k = (A_{ij}^k)_{m\times n}$ $(k = 1, 2, \cdots, t)$, the attribute weights and expert weights as well.

Proposed decision making method: In sum, an algorithm and process of the MAGDM problems based on TIT2FSs may be given as follows:

Step 1: Normalize the TIT2FS matrix $\tilde{A}^k = (A_{ij}^k)_{m\times n}$

In general, attributes can be classified into two types: Benefit attributes and cost attributes. In other words, the attribute set F can be divided into two subsets: F_1 and F_2, which are the subsets of benefit attributes and cost attributes, respectively. Since the n attributes may be measured in different ways, so the matrix $\tilde{A}^k = (A_{ij}^k)_{m\times n}$ needs to be normalized into $\tilde{R}^k = (R_{ij}^k)_{m\times n}$, where:

$$R_{ij}^k = (R_{ij}^{kU}, R_{ij}^{kL}) = ((r_{ij1}^{kU}, r_{ij2}^{kU}, r_{ij3}^{kU}, r_{ij4}^{kU}; H_1(R_{ij}^{kU}), H_2(R_{ij}^{kU})), (r_{ij1}^{kL}, r_{ij2}^{kL}, r_{ij3}^{kL}, r_{ij4}^{kL}; H_1(R_{ij}^{kL}), H_2(R_{ij}^{kL})))$$

For $j \in F_1$:

$$\begin{aligned}
&r_{ij1}^{kU} = \frac{a_{ij1}^{kU}}{\max_i\{a_{ij4}^{kU}\}},\; r_{ij2}^{kU} = \frac{a_{ij2}^{kU}}{\max_i\{a_{ij3}^{kU}\}},\; r_{ij3}^{kU} = \frac{a_{ij3}^{kU}}{\max_i\{a_{ij2}^{kU}\}} \wedge 1,\\
&r_{ij4}^{kU} = \frac{a_{ij4}^{kU}}{\max_i\{a_{ij1}^{kU}\}} \wedge 1,\; r_{ij1}^{kL} = \frac{a_{ij1}^{kL}}{\max_i\{a_{ij4}^{kL}\}},\; r_{ij2}^{kL} = \frac{a_{ij2}^{kL}}{\max_i\{a_{ij3}^{kL}\}},\\
&r_{ij3}^{kL} = \frac{a_{ij3}^{kL}}{\max_i\{a_{ij2}^{kL}\}} \wedge 1,\; r_{ij4}^{kL} = \frac{a_{ij4}^{kL}}{\max_i\{a_{ij1}^{kL}\}} \wedge 1
\end{aligned} \tag{9}$$

For $j \in F_2$:

$$\begin{aligned}
&r_{ij1}^{kU} = \frac{\min_i\{a_{ij1}^{kU}\}}{a_{ij4}^{kU}},\; r_{ij2}^{kU} = \frac{\min_i\{a_{ij2}^{kU}\}}{a_{ij3}^{kU}},\; r_{ij3}^{kU} = \frac{\min_i\{a_{ij3}^{kU}\}}{a_{ij2}^{kU}} \wedge 1,\\
&r_{ij4}^{kU} = \frac{\min_i\{a_{ij4}^{kU}\}}{a_{ij1}^{kU}} \wedge 1,\; r_{ij1}^{kL} = \frac{\min_i\{a_{ij1}^{kL}\}}{a_{ij4}^{kL}},\; r_{ij2}^{kL} = \frac{\min_i\{a_{ij2}^{kL}\}}{a_{ij3}^{kL}},\\
&r_{ij3}^{kL} = \frac{\min_i\{a_{ij3}^{kL}\}}{a_{ij2}^{kL}} \wedge 1,\; r_{ij4}^{kL} = \frac{\min_i\{a_{ij4}^{kL}\}}{a_{ij1}^{kL}} \wedge 1
\end{aligned} \tag{10}$$

The normalization method mentioned above is to preserve the property that the ranges of the normalized trapezoidal fuzzy numbers $[r_{ij1}^{kU}, r_{ij2}^{kU}, r_{ij3}^{kU}, r_{ij4}^{kU}]$ and $[r_{ij1}^{kL}, r_{ij2}^{kL}, r_{ij3}^{kL}, r_{ij4}^{kL}]$ belong to the closed interval (0, 1).

Step 2: By using the TIT2FS-WA operator to integrate all the attribute weights given by the experts, the weight of attribute a_j is obtained as follows:

$$\begin{aligned}
W_j = &\phi_\lambda(W_j^1, W_j^2, \cdots, W_j^t) = ((\sum_{k=1}^t \lambda_k w_{j1}^{kU}, \sum_{k=1}^t \lambda_k w_{j2}^{kU},\\
&\sum_{k=1}^t \lambda_k w_{j3}^{kU}, \sum_{k=1}^t \lambda_k w_{j4}^{kU}; \wedge_k H_1(W_j^{kU}), \wedge_k H_2(W_j^{kU})),\\
&(\sum_{k=1}^t \lambda_k w_{j1}^{kL}, \sum_{k=1}^t \lambda_k w_{j2}^{kL}, \sum_{k=1}^t \lambda_k w_{j3}^{kL}, \sum_{k=1}^t \lambda_k w_{j4}^{kL}; \wedge_k\\
&H_1(W_j^{kL}), \wedge_k H_2(W_j^{kL}))\quad (j = 1, 2, \cdots, n)
\end{aligned} \tag{11}$$

Step 3: Combined the weight vector of attributes obtained above and the ETIT2FS-WA operator to integrate all the attribute values of alternative A_i, the individual comprehensive value of alternative A_i is derived as follows:

$$\begin{aligned}
A_i^k = &\Gamma_W(A_{i1}^k, A_{i2}^k, \cdots, A_{in}^k) = ((\sum_{j=1}^n W_j r_{ij1}^{kU}, \sum_{j=1}^n W_j r_{ij2}^{kU},\\
&\sum_{j=1}^n W_j r_{ij3}^{kU}, \sum_{j=1}^n W_j r_{ij4}^{kU}; \wedge_j H_1(R_{ij}^{kU}), \wedge_j H_2(R_{ij}^{kU})),\\
&(\sum_{j=1}^n W_j r_{ij1}^{kL}, \sum_{j=1}^n W_j r_{ij2}^{kL}, \sum_{j=1}^n W_j r_{ij3}^{kL}, \sum_{j=1}^n W_j r_{ij4}^{kL}; \wedge_j\\
&H_1(R_{ij}^{kL}), \wedge_j H_2(R_{ij}^{kL})))\quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, t)
\end{aligned} \tag{12}$$

Step 4: Compute the collective overall attribute values of alternatives for the group. According to the method of the fuzzy linguistic quantifier (Xu, 2005), the correlated weighted vector w = $(w_1, w_2,..., w_t)^T$ with TIT2FS-HWA operator can be obtained. The collective comprehensive value of each alternative A_i is generated by Eq. 6 as follows:

$$A_i = \Phi_{w,\lambda}(A_i^1, A_i^2, \cdots, A_i^t) = \sum_{k=1}^t w_k A_i'^{\sigma(k)}\quad (i = 1, 2, \cdots, m) \tag{13}$$

where $\lambda = (\lambda_1, \lambda_2,..., \lambda_t)^T$ is the weight vector of experts, $A_i'^{\sigma(k)}$ is the kth largest TIT2FS of TIT2FSs $A_i'^k$ (i = 1, 2,..., n) with $A_i'^k = t\lambda_k A_i^k$.

Step 5: According to the collective overall comprehensive value of each alternative A_i, (i = 1, 2,..., m), the ranking order of alternatives is generated by the ranking method based on barycenter in section 2.

APPLICATION TO TEACHING QUALITY ASSESSMENT

A teaching quality assessment problem and the analysis process: In this section, the proposed MAGDM method is illustrated with a teaching quality assessment problem.

Table 1 shows the linguistic terms: Very Low (VL), Low (L), Medium Low (ML), Medium (M), Medium High (MH), High (H), Very High (VH) and their corresponding

Table 1: Linguistic terms and corresponding type 2 fuzzy sets (20)

Linguistic terms	Type 2 fuzzy sets
Very low (VL)	((0, 0, 0, 0.1; 1, 1), (0, 0, 0, 0.05; 0.9, 0.9))
Low (L)	((0, 0.1, 0.1, 0.3; 1, 1), (0.05, 0.1, 0.1, 0.2; 0.9, 0.9))
Medium low (ML)	((0.1, 0.3, 0.3, 0.5; 1, 1), (0.2, 0.3, 0.3, 0.4; 0.9, 0.9))
Medium (M)	((0.3, 0.5, 0.5, 0.7; 1, 1), (0.4, 0.5, 0.5, 0.6; 0.9, 0.9))
Medium High (MH)	((0.5, 0.7, 0.7, 0.9; 1, 1), (0.6, 0.7, 0.7, 0.8; 0.9, 0.9))
High (H)	((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 0.95; 0.9, 0.9))
Very High (VH)	((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))

Table 2: Weights of the attributes evaluated by the experts

Attributes experts	d_1	d_2	d_3
Safety	VH	H	VH
Price	H	VH	VH
Appearance	M	MH	MH
Performance	VH	H	H

interval type 2 fuzzy sets, respectively. Assume that there are three experts $D = \{d_1, d_2, d_3\}$ to evaluate three young teachers for their teaching quality, i.e., three teachers are formulated as the set of alternatives $A = \{A_1, A_2, A_3\}$. They will be evaluated by four attributes $F = \{a_1, a_2, a_3, a_4\}$, there are the teaching effect a_1, teaching method a_2, teaching designing a_3 and teaching content a_4, respectively. Suppose that the weight vector of experts is $\lambda = (0.3, 0.5, 0.2)^T$.

Assume that the three experts use the linguistic terms shown in Table 1 to represent the weights of the four attributes, respectively, as shown in Table 2. Assume that the three experts use the linguistic terms shown in Table 1 to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown in Table 3.

We solve the teaching quality assessment problem using the proposed method. The solving process is summarized as follows:

Step 1: According to Eq. 7 and 8, the normalized decision matrices are obtained shown in Table 4-6

Step 2: By using the TIT2FS-WA operator to integrate all the attribute weights given by the experts, the weight of attribute a_j is obtained as follows:

$$W_1 = ((0.80, 0.95, 0.95, 1.00; 1.00, 1.00), (0.875, 0.95, 0.95, 0.975; 0.90, 0.90))$$

$$W_2 = ((0.84, 0.97, 0.97, 1.00; 1.00, 1.00), (0.905, 0.97, 0.97, 0.985; 0.90, 0.90))$$

$$W_3 = ((0.44, 0.64, 0.64, 0.84; 1.00, 1.00), (0.54, 0.64, 0.64, 0.74; 0.90, 0.90))$$

$$W_4 = ((0.76, 0.93, 0.93, 1.00; 1.00, 1.00), (0.845, 0.93, 0.93, 0.955; 0.90, 0.90))$$

Table 3: Evaluating values of the alternatives given by the experts with respect to different attributes

Attributes	Alternatives decision-makers	d_1	d_2	d_3
a_1	x_1	MH	H	MH
	x_2	H	MH	H
	x_3	V	H	MH
a_2	x_1	H	VH	H
	x_2	MH	H	VH
	x_3	VH	VH	H
a_3	x_1	VH	H	H
	x_2	H	VH	VH
	x_3	M	MH	MH
a_4	x_1	VH	H	H
	x_2	H	VH	H
	x_3	H	VH	VH

Step 3: Combined the obtained weight vector of attributes and the ETIT2FS-WA operator to integrate all the attribute values of alternative A_i, the individual comprehensive value of each alternative is got as follows:

$$A^1_1 = ((1.90, 2.9895, 2.9895, 3.8400; 1.0, 1.0), (2.4127, 2.9895, 2.9895, 3.5109; 0.9, 0.9))$$

$$A^1_2 = (1.867, 3.0225, 3.0225, 3.840; 1.0, 1.0), (2.4868, 3.2380, 3.2380, 3.6162; 0.9, 0.9))$$

$$A^1_3 = ((1.804, 2.786, 2.786, 3.6534; 1.0, 1.0), (2.2663, 2.7860, 2.786, 3.2370; 0.9, 0.9))$$

$$A_1^2 = ((2.1073, 3.333, 3.3330, 3.84; 1.0, 1.0), (2.5688, 3.2360, 3.236, 3.6650; 0.9, 0.9))$$

$$A_2^2 = ((2.068, 3.2789, 3.2789, 3.84; 1.0, 1.0), (2.6308, 3.2789, 3.2789, 3.665; 0.9, 0.9))$$

$$A_3^2 = ((2.1713, 3.298, 3.298, 3.6534; 1.0, 1.0), (2.682, 3.2010, 3.201, 3.5481; 0.9, 0.9))$$

$$A_1^3 = ((1.828, 3.1219, 3.1219, 3.84; 1.0, 1.0), (2.4230, 3.1219, 3.1219, 3.665; 0.9, 0.9))$$

$$A_2^3 = ((2.1896, 3.30, 3.3000, 3.84; 1.0, 1.0), (2.6130, 3.205, 3.205, 3.6162; 0.9, 0.9))$$

Table 4: Normalized decision matrix given by expert d_1

d_1	a_1	a_2
x_1	((0.5, 0.7, 0.7,1;1,1), (0.6, 0.7, 0.7, 0.84; 0.9 0.9))	((0.5, 0.778, 0.778, 1;1,1), (0.632, 0.778, 0.778, 1; 0.9,0.9))
x_2	((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 0.95; 0.9, 0.9))	(0.5556, 0.778, 0.778, 1; 1, 1), (0.75, 1, 1, 1; 0.9, 0.9))
x_3	((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))	((0.5, 0.7, 0.7, 1; 1, 1), (0.6, 0.7, 0.7, 0.842; 0.9, 0.9))
d_1	a_3	a_4
x_1	((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))	((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))
x_2	((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9))	((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9))
x_3	((0.3, 0.5, 0.5, 0.7778; 1, 1), (0.4, 0.5, 0.5, 0.632; 0.9, 0.9))	((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9))

Table 5: Normalized decision matrix given by expert d_2

d_1	a_1	a_2
x_1	(0.7, 1, 1, 1; 1, 1), (0.842, 1, 1, 1; 0.9, 0.9)	(0.842, 1, 1, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
x_2	(0.5, 0.778, 0.778, 1; 1, 1), (0.632, 0.778, 0.778, 1; 0.9, 0.9)	(0.7, 1, 1, 1; 1, 1), (0.842, 1, 1, 1; 0.9, 0.9)
x_3	(0.7, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)	(0.842, 1, 1, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
d_1	a_3	a_4
x_1	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
x_2	(0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)	(0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)
x_3	(0.5, 0.7, 0.7, 0.778; 1, 1), (0.6, 0.7, 0.7, 0.842; 0.9, 0.9)	(0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)

Table 6: Normalized decision matrix given by expert d_1

d_1	a_1	a_2
x_1	(0.5, 0.778, 0.778, 1; 1, 1), (0.632, 0.778, 0.778, 1; 0.9, 0.9)	(0.7, 1, 1, 1; 1, 1), (0.84, 1, 1, 1; 0.9, 0.9)
x_2	(0.842, 1, 1, 1; 1, 1), (0.8, 0.9, 0.9, 0.95; 0.9, 0.9)	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
x_3	(0.632, 0.778, 0.778, 1; 1, 1), (0.95, 1,1,1; 0.9, 0.9)	(0.842, 1,1,1; 1,1), (0.6, 0.7, 0.7, 0.842; 0.9, 0.9)
d_1	a_3	a_4
x_1	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
x_2	(0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)	(0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 1; 0.9, 0.9)
x_3	(0.5, 0.7, 0.7, 0.9; 1, 1), (0.6, 0.7, 0.7, 0.842; 0.9, 0.9)	(0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9)

$$A_3^3 = ((2.1169, 3.0869, 3.0869, 3.756; 1.0, 1.0), (2.5010, 3.007, 3.007, 3.3924; 0.9, 0.9))$$

Step 4: According to the method of the fuzzy linguistic quantifier (Xu, 2005), we choose the "almost" criteria and determine that the correlated weighted vector $w = (w_1, w_2,..., w_t)^T$ with TIT2FS-HWA operator is W = (0.067, 0.667, 0.267). Combined the weight vector of experts $\lambda = (0.3, 0.5, 0.2)^T$, the collective comprehensive value of the alternative A_i is derived by Eq. 13 as follows:

$$A_1 = ((1.7916, 2.8797, 2.8797, 3.6138; 1.0, 1.0), (2.2888, 2.87, 2.8700, 3.3567; 0.9, 0.9))$$

$$A_2 = ((1.9708, 3.0368, 3.0368, 3.6138; 1.0, 1.0), (2.4305, 3.032, 3.032, 3.4082; 0.9, 0.9))$$

$$A_1 = ((1.9225, 2.854, 2.854, 3.4998; 1.0, 1.0), (2.3155, 2.7963, 2.7963, 3.1709; 0.9, 0.9)).$$

Step 5: By the method of section 2, the coordinates of barycenters for A_1, A_2 and A_3 are obtained as follows:

$$M_1 = 0.4305, M_{1c} = 0.1595, M_{1y} = 1.1521, x_1 = 2.6760, y_1 = 0.3705$$

$$M_2 = 0.3815, M_{2x} = 0.3137, M_{2y} = 1.0560, x_2 = 2.7781, y_2 = 0.8221$$

$$M_3 = 0.4037, M_{3x} = 0.1474, M_{3y} = 1.1129, x_x = 2.7567, y_3 = 0.3651$$

By Eq. 1, the ranking values are generated as follows:

$$R(A_1) = 0.9916, R(A_2) = 2.2840, R(A_3) = 1.0065$$

Since, $R(A_3)>R(A_2)>R(A_1)$, the ranking order is $A_2 \succ A_3 \succ A_1$, the best teacher is A_2.

Comparison analysis with similar methods: Chen and Lee (2010b) extended the TOPSIS method to solve the MAGDM problem based on interval type 2 fuzzy set. The proposed method in this study uses the arithmetic average operators on TIT2FSs to aggregate the attribute values of alternatives and the attribute weight information and then ranks the alternatives. If we use the method of Chen and Lee (2010b) to solve the above teaching quality assessment problem, the ranking order obtained by the method of Chen and Lee (2010a, b) $A_2 \succ A_3 \succ A_1$, which is just the same as that obtained by the method proposed in this study. This analysis verifies the effectiveness and validation of the proposed method in this study. Compared with Lee and Chen (2008b) and Chen and Lee (2010a, b), this study has some differences and advantages, which are summarized as follows:

- This study sufficiently considers the importance degrees of different experts. On the one hand, the attribute weight is obtained by combining the TIT2FS-WA operator and the expert weights. While Lee and Chen (2008b) and Chen and Lee(2010b) only adopted the equal weights of experts to derive the attribute weights through the simple arithmetic average. On the other hand, before utilizing the TIT2FS-HWA operator in this study, the individual comprehensive values of alternatives should first be weighted by the expert weights and then the collective comprehensive values of alternatives can be got
- Lee and Chen (2008b) and Chen and Lee (2010b) didn't consider the importance degrees of different experts at all. In fact, different experts act as different roles in the process of MAGDM. Consider that some decision makers may assign unduly high or unduly low uncertain preference values to their preferred or repugnant objects, to relieve the influence of these unfair arguments on the decision results and to reflect the importance degrees of all the experts, the proposed method in this study first weights each individual comprehensive value by using the corresponding expert weights and then utilizes the TIT2FS-HWA operator to aggregate all the individual weighted comprehensive values into the collective comprehensive value for each alternative. Therefore, the TIT2FS-HWA operator can assign low weights to those "false" or "biased" arguments and make the decision results more reasonable. These advantages can not be reflected in the method of Chen and Lee (2010b)
- The ranking value of TIT2FSs developed by Chen and Lee (2010a) is not always reasonable since they defined the strength of trapezoidal membership functions may be negative, whereas this study proposes a new ranking method based on barycenter which has intuitive and geometrical meaning is more reasonable

CONCLUSION

This study presents a new ranking method of interval type 2 fuzzy sets based on barycenter and develops the TIT2FSs weighted arithmetic average operator, ordered weighted average operator, hybrid weighted average operator and extended TIT2FSs weighted average operator for the first time.

For the MAGDM problem based on interval type 2 fuzzy sets, the weight vector of attributes is obtained by combining the TIT2FS-WA operator and the weight vector of experts. Combining the weight vector of attributes and the ETIT2FS-WA operator, the individual comprehensive value of alternative is then derived. Using the TIT2FS-HWA operator, the individual comprehensive values of alternatives are integrated into the collective comprehensive value of alternative. According to the collective comprehensive values of alternatives, a new method of MAGDM based on interval type 2 fuzzy sets is thus proposed. This method sufficiently considers the importance degrees of different experts, which can make the decision results more reasonable.

Although, the proposed method is illustrated with the teaching quality assessment problem, it is expected to be applicable to real-life decision problems in many areas such as risk investment, performance evaluation of military system, engineering management, supply chain and so on. In the near future, we will investigate some geometric aggregation operators with interval type 2 fuzzy sets and apply them to the fuzzy MAGDM problems.

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