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Comparisons for Three Kinds of Quantile-based Risk Measures

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Abstract: This article compares three kinds of quantile-based risk measures: VaR, ES and a new proposed coherent risk measure called iso-entropic risk measure. The main factors to be compared are convexity, the volume of information which is used to measure the risk, relationship between these risk measures and stochastic dominances. It is pointed that though ES holds convexity, it only utilizes local information as VaR and is consistent with stochastic dominances lowers than second-order. However, iso-entropic risk measure utilizes the whole information to measure the risk, it is not a 0-1 risk measure and it is consistent with stochastic dominances of almost all the orders. So, it is most powerful for discrimination of risk. Simulation cases demonstrate this.

Key words: Risk management, iso-entropic risk measure, coherent risk measure

INTRODUCTION

How to measure the risk of a position is one of the basic tasks in finance. The most widely used methods in practice are variance and subsequent VaR (Value at Risk). However, both have serious drawbacks. One important drawback for variance is that it is not monotonic: a gamble with higher gains and lower losses may have a higher variance and thus be wrongly viewed as having a higher riskiness. About VaR, it concerns only with the probability of the loss and does not care about the size of the loss. However, it is obvious that the size of loss should be taken into account (Cherny and Madan, 2008). Further criticism of variance and VaR can be found in Artzner *et al.* (1997) as well as in numerous discussions in financial journals.

At the final of last century, a new promising method to quantify risk was proposed in the landmark paper by Artzner *et al.* (1999). That is coherent risk measure. And later, the coherent risk measure was extended to the class of convex risk measures (Follmer and Schied, 2002, 2008; Frittelli and Gianin, 2002; Heath, 2000). Since, their seminal work, the theory of coherent risk measures has rapidly been evolving (Acerbi and Tasche, 2002; Artzner *et al.*, 2007; Ben-Tal and Teboulle, 2007; Cherny, 2006; Delbaen, 2002; Fischer, 2003; Heath and Ku 2004; Jaschke and Kuchler, 2001; Jouini *et al.*, 2004; Leitner, 2004; Rockafellar and Uryasev, 2002; Tasche, 2002). Excellent reviews on the theory of coherent risk measure is given in Follmer and Schied (2004). Recently, the most fashionable coherent risk measure is ES (Expected Shortfall, ES thereafter). Compared to VaR, it measures not only the probability of loss but its severity as well. Kusuoka (2001) proved that ES is the smallest law invariant coherent risk measure that dominates VaR. It is seemly that ES might be the most important subclass of coherent risk measures. However, its disadvantage is that it depends only on the tail of the distribution, so it is not smooth (Cherny and Madan, 2008). Zheng and Chen (2012) proposed a new coherent risk measure based on relative entropy which is obtained under the theory framework of coherent risk measure from Artzner *et al.* (1999). We call this new measure iso-entropic risk measure. Our article compares the three measures: VaR, ES and iso-entropic risk measure. Because they can be represented with quantiles, they are called quantile-based measures. The mainly factors to be compared are convexity, the volume of information which is used to measure the risk, relationship between these risk measures and stochastic dominances. And the results are demonstrated with simulated cases.

Quantile-based risk measures: Here, we introduce three kinds of quantile-based risk measures. Especially, one new coherent risk measure is proposed based on the given relative entropy. The first one is VaR. at level a∈(0, 1), VaR defined for X on a probability space (Ω, F, P) is:

$$VaR_{\alpha}(X) = \inf\{m\in R/P[X+m<0]\}\leq\alpha \qquad (1)$$

From the Eq. 1, it is seen that the VaR is quantile-based, because $VaR_{\alpha}(X) = -q_{\alpha}(X)$ where, $q_{\alpha}(X)$ is the α-quantile of X. VaR satisfies monotonicty, translation

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invariance and positive homogeneity but not subadditivity, it is not convex.

The second one is ES. At level $\alpha\in(0,1)$, ES is defined as:

$$ES_\alpha(X)=\frac{1}{\alpha}\int_0^\alpha VaR_u(X)du=\frac{1}{\alpha}\int_0^\alpha -q_u(X)du \quad (2)$$

ES is also called Conditional Value at Risk (CVaR) and Average Value at Risk (AVaR). ES is coherent risk measure according to the basic representation theorem proved by Artzner *et al*. (1999). It satisfies all the four axioms: monotonicity, translation invariance, positive homogeneity and subadditivity. From Eq. 2, we know that it depends only on the tail of the distribution, i.e. it is a 0-1 risk measure, so it is not smooth.

The third one is proposed by Zheng and Chen (2012), which is called iso-entropic risk measure. The measure is induced through the basic representation theorem of coherent risk measure proposed by Artzner *et al*. (1999) for a finite Ω and by Delbaen (2002) in the general case. The iso-entropic risk measure is:

$$\rho_\alpha(X) = -E_P[Z^*X) \quad (3)$$

where:

$$Z^*=z(X,\alpha)=\frac{e^{-mX}}{c}$$

$E[^{-mX}]$, $E[\bullet] = E_P[\bullet]$, the subscript is omitted in the sequel. And $m\geq 0$, it satisfies the following equation:

$$E\left[\frac{(-mX-\log c)e^{-mX}}{c}\right]=H_\alpha \quad (4)$$

Here, m = (X,α) is determined by x and α through Eq. 4 uniquely. Apparently, we can rewrite it based on quantile:

$$\rho_\alpha(X)=\int_0^1 -q_u(X)z_u(X,\alpha)du \quad (5)$$

where:

$$z_u(X,\alpha)=\frac{e^{-mq_u(X)}}{c}$$

Apparently, iso-entropic risk measure is coherent. It satisfies all the four axioms.

Comparison for three risk measures: Here, we compare the three quantile-based risk measures in several facets. Firstly, we consider the convexity of these risk measures. According to the four axioms, VaR does not has the convexity but ES and iso-entropic risk measure have the convexity. Coherence means convexity. Convexity is a very good character in finance. Acerbi (2003) and Krokhmal *et al*. (2011) pointed that convexity helps the optimization of investment portfolio.

Secondly, we consider the volume of information which is used to measure the risk. For the quantile-based risk measures, a unified expression can be given as follows:

$$R_\alpha(X)=-\int_0^1 q_t(X)\phi_t(X,\alpha)dt \quad (6)$$

For VaR, $\phi(X,\alpha)=\delta_{t=\alpha}$ where, $\delta_{t=\alpha}$ is Dirac function. And for ES, $\phi_t(X,\alpha)=\frac{1}{\alpha}1_{t\leq\alpha}$ bur for iso-entropic risk measure. $\phi_t(X,\alpha)=\frac{e^{-mq_t(X)}}{c}$ From these, we can see that the VaR only reflects information of a single quantile, ES reflects information of quantiles where, $t\leq\alpha$ both are local information of the distribution. However, the iso-entropic risk measure reflects information of the whole distribution of risk X.

Lastly, we consider the relationship between these risk measures and stochastic dominances in the next section.

Stochastic dominances and risk measures: Here, we will show the linkage between the stochastic dominances and the three quantile-based risk measures.

Stochastic dominance: Firstly, we recall the definitions about stochastic dominances. X first-order stochastically dominates Y if $F_X(\eta)\leq F_Y(\eta)$ for all $\eta\in R$; Generally, denote $F_X^{(n)}(\eta)=\int_{-\infty}^{\eta}F_X^{(n-1)}(\xi)d\xi$ then n-order stochastically dominates Y if:

$$F_X^{(n)}(\eta)\leq F_Y^{(n)}(\eta)\leq. \text{ For all } \eta\in R,\ n = 2, 3...$$

The first-order stochastic dominance is known for its equivalence for unanimous choices made by investors with monotonic expected utility functions while the second one is known for its equivalence for unanimous choices made by investors with risk-averse expected, utility investors. For notational simplicity, we write $X\overset{FSD}{\succeq}Y$, $X\overset{SSD}{\succeq}Y$ and $X\overset{nSD}{\succeq}Y$ whenever Y stochastically dominates Y according to FSD, SSD and nSD (first, second and n-order stochastic dominance), respectively.

VaR, ES and stochastic dominance: According to Ma and Wong (2010) and Wong and Ma (2008), VaR is only consistent with the first-order stochastic dominances.

This means: for all X and Y, the following statements are right:

$$X \overset{FSD}{\succeq} Y \Leftrightarrow \text{VaR}_\alpha(X) \le \text{VaR}_\alpha(Y), \forall \alpha \in (0,1] \tag{7}$$

ES is consistent with the first-order stochastic dominance, namely the following statements is hold:

$$X \overset{FSD}{\succeq} Y \Rightarrow \text{ES}_\alpha(X) \le \text{ES}_\alpha(Y), \forall \alpha \in (0,1] \tag{8}$$

ES is consistent with second-order stochastic dominance; moreover, the following statements are right:

$$X \overset{SSD}{\succeq} Y \Leftrightarrow \text{ES}_\alpha(X) \le \text{ES}_\alpha(Y), \forall \alpha \in (0,1] \tag{9}$$

However, ES is not consistent with the other order stochastic dominances.

Iso-entropic risk measure and stochastic dominance: Similarly, iso-entropic risk measure is consistent with the first-order stochastic dominance. We have the following Theorem:

Theorem 1: For all X and Y that are absolutely integrable, it must hold true that:

$$X \overset{FSD}{\succeq} Y \Rightarrow \rho_\alpha(X) \le \rho_\alpha(Y), \forall \alpha \in (0,1] \tag{10}$$

It is apparent. In fact, this character is equivalent to the monotonicity condition of coherent risk measure.

For SSD, iso-entropic risk measure is consistent with it, too. We have the following theorem.

Theorem 2: For all X and Y that have continum supports and are atomless and absolute integrable, it must hold true that:

$$X \overset{SSD}{\succeq} Y \Rightarrow \rho_\alpha(X) \le \rho_\alpha(Y), \forall \alpha \in (0,1] \tag{11}$$

The proof is as follows:

$$\begin{aligned}\rho_\alpha(X) &= \int_0^1 -q_t(X) z_t(X,\alpha)\,dt \\ &= \int_0^1 -q_t(X)\left[\int_t^1 -z'_u(X,\alpha)\,du\right]dt \\ &= \int_0^1 -uz'_u(X,\alpha)\left[\frac{1}{u}\int_0^u -q_t(X)\,dt\right]du \\ &= \int_0^1 -uz'_u(X,\alpha)\text{ES}_u(X)\,du\end{aligned}$$

From the representation theorem of itself, we have:

$$\begin{aligned}\rho_\alpha(Y) &\ge \int_0^1 -z_t(X,\alpha) q_t(Y)\,dt \\ &= \int_0^1 -q_t(Y)\left[\int_t^1 -z'_u(X,\alpha)\,du\right]dt \\ &= \int_0^1 -uz'_u(X,\alpha)\text{ES}_u(Y)\,du\end{aligned}$$

So we get:

$$\rho_\alpha(Y) - \rho_\alpha(X) \ge \int_0^1 -uz'_u(X,\alpha)(\text{ES}_u(Y) - \text{ES}_u(X))du$$

Here:

$$z'_u(X,\alpha) = -\frac{me^{-mq_u(X)}}{c} q'_u(X)$$

And:

$$q'_u(X) = \frac{1}{f_X(q_u(X))} \ge 0$$

So, $-uz'_u(X, \alpha) \ge 0$. Then utilizing the Eq. 9, we get the result. The proof is completed.

The inverse direction of Eq. 11 is not true. Supposed that:

$$\int_0^1 -uz'(X,\alpha)(\text{ES}_t(Y) - \text{ES}_t(X))dt \ge 0$$

We can not conclude that $\text{Es}_t(Y)-\text{Es}_t(X)>0$.

In general, iso-entropic risk measure is consistent with n-order stochastic dominance. We have the following Theorem:

Theorem 3: For all X and Y that have continum supports and are atomless and absolute integrable, then for $\forall \alpha \in (0, 1)$, it must hold true that:

$$X \overset{nSD}{\succeq} Y \Rightarrow \rho_\alpha(X) \le \rho_\alpha(Y),\ n = 2,... \tag{12}$$

The proof is as follows.

When n = 2, Eq. 12 holds true. Now, assume that for n>2, Eq. 12 holds true.

From Eq. 9 and Eq. 11, we have:

$$\begin{aligned}\rho_\alpha(Y) - \rho_\alpha(X) \ge& \int_0^1 -z'_u(X,\alpha) F_Y^{(2)}(q_u(X))\,du \\ &- \int_0^1 -z'_u(X,\alpha) F_X^{(2)}(q_u(X))\,du\end{aligned}$$

Apparently:

$$
\begin{aligned}
&\int_0^1 -z'_u(X,\alpha)F_X^{(2)}(q_u(X))du \\
&= \int_0^1 F_X^{(2)}(q_u(X))q'_u(X)\left[\int_u^1 \left(\frac{z'_t(X,\alpha)}{q'_t(X)}\right)' dt\right]du \\
&= \int_0^1 -m(X,\alpha)z'_t(X,\alpha)F_X^{(3)}(q_t(X))dt \\
&\ldots \\
&= \int_0^1 -(m(X,\alpha))^{n-2} z'_t(X,\alpha)F_X^{(n)}(q_t(X))dt
\end{aligned}
$$

And:

$$
\begin{aligned}
&\int_0^1 -z'_u(X,\alpha)F_Y^{(2)}(q_u(X))du \\
&= \int_0^1 -(m(X,\alpha))^{n-2} z'_t(X,\alpha)F_Y^{(n)}(q_t(X))dt
\end{aligned}
$$

This means that:

$$
\begin{aligned}
&\rho_\alpha(Y)-\rho_\alpha(X) \geq \\
&\quad \int_0^1 h(\bullet)\left(F_Y^{(n)}(q_t(Y))-F_X^{(n)}(q_t(Y))\right)dt
\end{aligned}
$$

where, h(•) So Eq. 12 holds true. Then the proof is completed.

The power of discrimination for risk measures: The order of stochastic dominance which risk measures is consistent with determines the power of discrimination for risk measures. Higher the order is, more powerful is discrimination for risk measure. To demonstrate this, we give the case of simulation in this section.

Here, we use three typical pairs of risks to test the power of discrimination. All the risks come from simulation of normal distribution and extreme value distribution. For example, we simulate 4000 draws from each distribution and then replace some draws from one distribution with the same quantity of draws from another distribution. All the parameters of distribution for simulation are based on the data from A300 stock index in China. The two mainly parameters used are mean and standard error of daily returns for A300 stock index. The data interval is 2005.04.08-2010.08.06. Some descriptive statistics for A300 stock index and the simulated risks are in Table 1.

In Table 1, all the descriptive statistics for pairs (a, b and c) are from the simulated data. In pair a, risk X is drawed from Gaussian distribution, X~N(μ, σ) where, parameters takes from A300(serves as the base), the left tail of 5% for risk Y is the same as risk X, the rest of 95% is drawed from.

Gaussian distribution Y~N(μ_1, σ_1) where, the two parameters has a subtle change. The empirical distribution for pair a is saw in the left subfigure in Fig. 1. In pair b, risk X is drawed from the same distribution as that in pair the right tail of 95% for risk Y is the same as risk X, the rest of 5% is drawed from Gaussian distribution Y~N(μ_2, σ_2) where, the two parameters has a subtle

Table 1: Some descriptive statistics for simulated pairs of risks

		Mean	Std	Skewness	Kurtosis
Base	A300	0.001	0.0213	-0.2900	5.10
Pair a	Risk X	0.0012	0.0217	-0.0030	3.14
	Risk Y	-0.001	0.021	0.0550	3.71
Pair b	Risk X	0.0008	0.0216	-0.0625	3.02
	Risk Y	0.0008	0.0216	-0.0745	3.08
Pair c	Risk X	-0.012	0.027	-1.2530	6.07
	Risk Y	-0.012	0.027	-1.2500	6.40

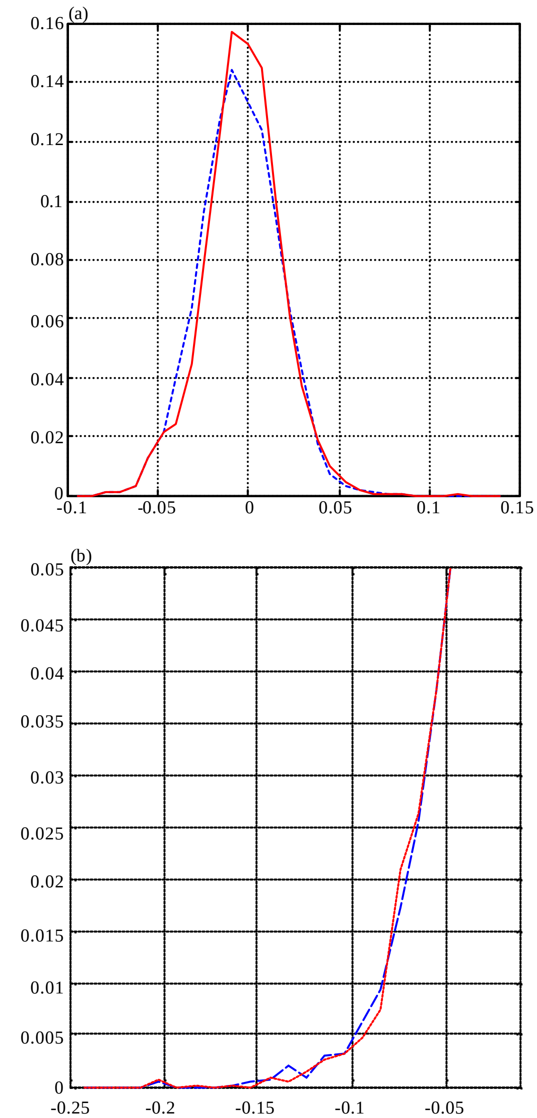

Fig. 1(a-b): Empirical distribution for simulated data (a) Whole distribution for pair a, (b) Left tail of 5% for pair b or c Rest 95% distribution for risk X and Y are the same

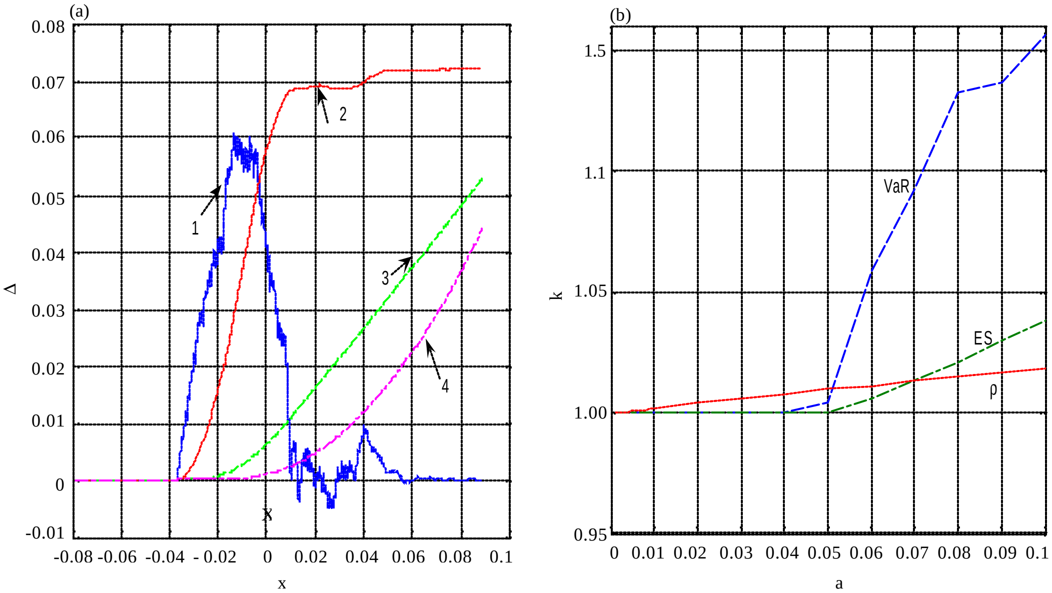

Fig. 2(a-b): Stochastic dominances and power of discrimination for pair a (a) Stochastic dominances and (b) Power of discrimination

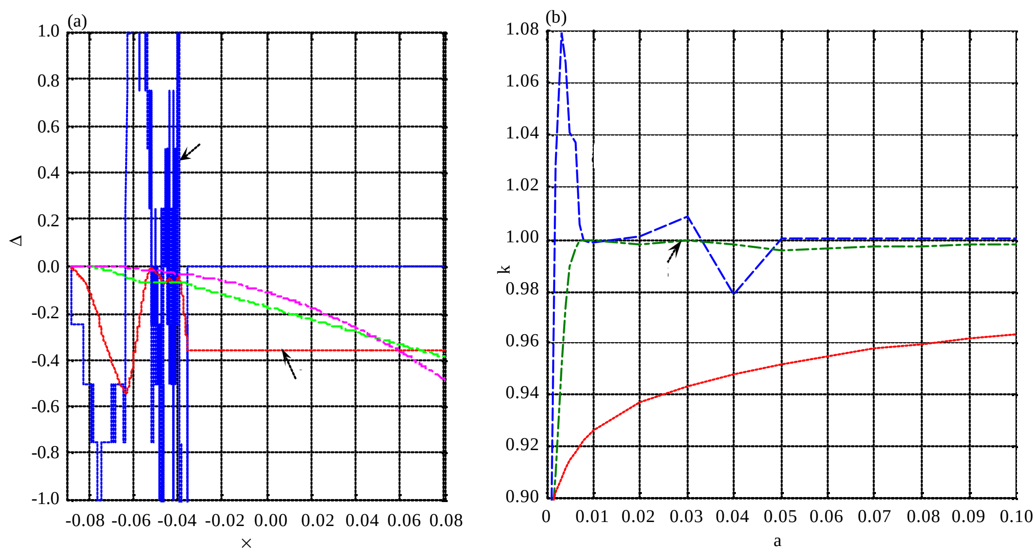

Fig. 3(a-b): Stochastic dominances and power of discrimination for pair b (a) Stochastic dominances and (b) Power of discrimination

change. The empirical distribution for pair b see the right subfigure in Fig. 1, here we only present the different left tail of 5%. The mechanism of simulation data for pair c is the same as pair b but the extreme value distribution takes a role when simulation data are produced. The empirical distribution for pair c is similar to pair b.

Now, we utilize these pairs of risks to test the power of discrimination of three kinds of quantile-based risk measures. Firstly, we consider the stochastic dominances for each pairs of risks. Denote $\Delta(n) = F_X^{(n)}(x)-F_Y^{(n)}(x)$. If $\Delta(n)\leq 0$, for all $x\in R$, $n = 1,2,3...$, then X is said to n-order stochastically dominates Y. Apparently, if X is said to n-order stochastically dominates Y, then X stochastically dominates Y at any order higher than n. And denote:

$$k = \frac{R_\alpha(Y)}{R_\alpha(X)}$$

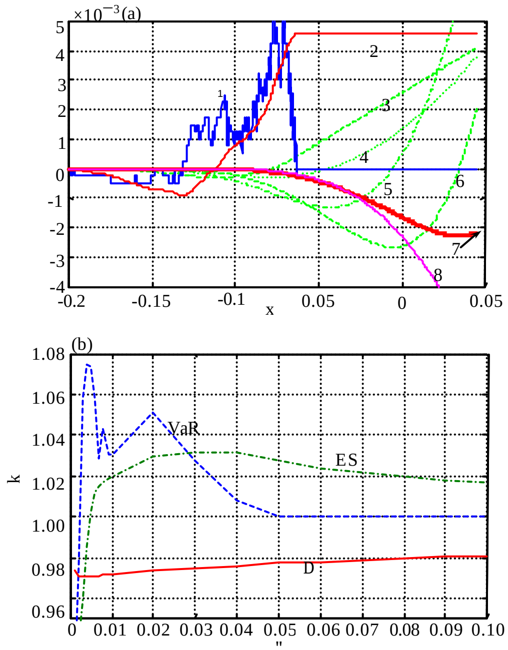

Fig. 4(a-b): Stochastic dominances and power of discrimination for pair c (a) Stochastic dominances (b) Power of discrimination

where, $R_\alpha(\bullet)$ is one of the three quantile-based risk measures considered in this study.

For pair a, the results for $\Delta(n)$ and k are in Fig. 2. From the figure, risk X second-order stochastically dominates X but do not first-order stochastically dominates Y. So, for ES and iso-entropic risk measure ρ_α, $k \geq 1$, at any level $\alpha \in (0,1)$ but for VaR, $k \geq 1$ only at some level. In Fig. 2 , we only give the level $\alpha \in (0, 0, 0, 1)$. Though, the left tail of 5% of distribution for risk X and Y are the same, $k>1$ for iso-entropic risk measure ρ_α. But $k = 1$ for VaR and ES. It indicates that the power of discrimination for iso-entropic risk measure may be higher than the other two measures.

The results for $\Delta(n)$ and k for pair b are in Fig. 3. We can see that risk Y second-order stochastically dominates X but do not first-order stochastically dominates X. It can be seen that at th most important level $\alpha \in (0, 0, 0.05)$, k is vibrating for VaR which indicates that the power of discrimination for VaR is lowest.

The results for $\Delta(n)$ and k for pair c are in Fig. 4. We can see that risk Y seventh-order stochastically dominates X but do not stochastically dominates X below seventh-order. It can be seen that at some level, k of VaR and ES is larger than one but at some level, k of VaR and ES is smaller than one. For iso-entropic risk measure $\rho\alpha$, k is always stable. So, it indicates that the power of discrimination for VaR is lowest and the power of discrimination for iso-entropic risk measure is highest. This is because that the volume of information which is used to measure the risk plays a key role.

CONCLUSION

This article compares several current important risk measures based on quantiles with a new proposed coherent risk measure called iso-entropic risk measure. The mainly factors to be compared are convexity, the volume of information which is used to measure the risk, relationship between these risk measures and stochastic dominances. Specially, we give the details about relationship between these risk measures and stochastic dominances and demonstrate our results with simulated cases. It turns out that the new risk measure has some advantage over the other two quantile-based risk measures, especially for consistency with stochastic dominances which provides powerful discrimination for risks. So, its characteristics and applications need to be exploited further.

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