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ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 13 (7): 1451-1454, 2014
ISSN 1812-5638 / DOI: 10.3923/itj.2014.1451.1454

The Periodicity and the Horseshoes of Continuous Self-maps

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Abstract: The continuous self-maps on completely densely ordered linear ordered topological spaces (shortly, CDOLOTS) are discussed. This study is the generalization of the ones on the real line. It is pointed that n-chain of intervals implies periodic points with period-n. Then, the following two results are obtained. (1) Period-3 implies arbitrary integer period. (2) Horseshoes implies arbitrary integer period. At last, Sufficient conditions of the existence for periodic points of odd period are obtained.

Key words: Periodic points, horseshoes, continuous self-maps, CDOLOTS

INTRODUCTION

In engineering applications such as digital communication principles, environmental science, biological science, encryption and energy, the periodicity of a system is an important subject for study. The properties of periodic points or periodic solutions are intensively discussed recently (Buica and Ortega, 2012; Kurkcu *et al*., 2011; Graff *et al*., 2010; Abbas and Rhoades, 2009). This study considers the periodicity of self-maps on CDOLOTSs.

The order relationship of linear ordered set X is denoted by $a \prec b$. Notation $a \prec = b$ denotes '$a \prec b$ or a = b'. Definitions and notations of interval in X are similar to the real line. Infinite intervals are denoted by $(\leftarrow, a]$, $[b, \rightarrow)$ and $(\leftarrow, \rightarrow)$. If there exists an element $m \in X$ such that $x \prec = m$ for all $x \in X$, m will be called for largest element of X and denoted by End {X}. If there exists an element $m \in X$ such that $m \prec = x$ for all $x \in X$, m will be called for smallest element of X and denoted by Ah{X}. If X is a topological space with the subbase $\{(a, \rightarrow), (\leftarrow, a): a \in X\}$, then X is called a linear ordered topological space. X is densely ordered if whenever $\{a, b \in X: a \prec b\}$ there is an element $c \in X$ such that $a \prec c \prec b$. X is complete if every nonempty set with upper bound has a least upper bound in X. X is called a completely densely ordered linear ordered topological spaces if it is a linear ordered topological space with complete and densely order relation.

Let X is a CDOLOTS, $f: X \rightarrow X$ a continuous map. If $\Delta_1, \Delta_2, \ldots, \Delta_n$ are closed subintervals of X such that:

$$\Delta_i \subset f(\Delta_{i-1}) \text{ for } 2 \le i \le n$$

then $\Delta_1, \Delta_2, \ldots, \Delta_n$ is called a n-chain of intervals for f. If $\Delta_1, \Delta_2, \ldots, \Delta_n$ are closed subintervals of X with pairwise disjoint interiors such that:

$$\Delta_1 \cup \Delta_2 \cup \ldots \cup_n \subset f(\Delta_i)$$

for every integer i from 1 to n, then $\Delta_1, \Delta_2, \ldots, \Delta_n$ is called a n-horseshoe, or simply a horseshoe if n = 2. Block and Coppel (1986) called a map with a horseshoe turbulent. Other concepts and terms (e.g., separation, connected) are from (Todorcevic, 1984; Nagatal, 1985). If X is a CDOLOTS, then X and integers in X is connected.

THE EXISTENCE OF PERIODIC POINTS

This section introduces the relations between chain of intervals and periodic points, horseshoes and periodic points. In this section, X is a CDOLOTS and $f: X \rightarrow X$ a continuous map.

Lemma: 1 If [a, b] and [c, d] are subintervals of X satisfied $f([a, b]) \supset [c, d]$, then there exists a closed interval Δ in [a, b] such that:

$$f(\Delta) = [c, d] \text{ and } f(\Delta') \neq [c, d]$$

for arbitrary proper subinterval Δ' of Δ.

Proof let:

$$U = \{x \in [a, b]: f(x) = d\} \neq \phi$$

$$V = \{x \in [a, b]: f(x) = c\} \neq \phi$$

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Without loss of generality, suppose $u \in U$, $v \in V$: $v \prec u$. Put:

$$u^* = \text{Ah}\{x: x \in U \cap [v, b]\}$$

$$v^* = \text{End}\{x: x \in V \cap [v, u^*]\}$$

For $\forall x \in (v^*, u^*)$, one has $f(x) \neq c$, $f(x) \neq d$ and $\Delta = [v^*, u^*] \subset [a, b]$.

Since $f(v^*) = c$, $f(u^*) = d$, one has $f([v^*, u^*] \subset [c, d])$.

On the other hand, hypothesis there exists a x in (v^*, u^*) such that $f(x) \notin [c, d]$. Suppose $d \prec d_1 = f(x)$ ($d_1 \prec d$) is similar). Since $d \in (c, d_1)$, there exists a point $x' \in (v^*, u^*)$ such that $f(x')$. This contradicts $f(x) \neq d$ for every $x \in (v^*, u^*)$. So:

$$f([v^*, u^*]) \subset [c, d]$$

Thus, $f(\Delta)$. By the definitions of u^* and v^*, one has $f(\Delta') \neq [c, d]$ for arbitrary proper subinterval Δ' of Δ.

Theorem 1: If $(\Delta_0, \Delta_1, ..., \Delta_{n-1})$ is a chain of intervals for f and $f(\Delta_{n-1}) \supset \Delta_0$, then there exists a point x_0 in Δ_0 such that $f^n(x_0) = \Delta_0$ and $f^n(x_0) \in \Delta_i$ for every integer i from 1 to n-1.

Proof: Since $f(\Delta_{n-1}), \supset \Delta_0$ there exists a closed subinterval Δ_{n-1}^* of Δ_{n-1} such that $f(\Delta_{n-1}^*)$. And because:

$$f(\Delta_{n-2}) \supset \Delta_{n-1} \supset \Delta_{n-1}^*$$

so there exists a closed subinterval Δ_{n-2}^* of Δ_{n-2} such that $f(\Delta_{n-2}^*) = \Delta_{n-1}^*$.

Similarly, there are closed subintervals:

$$\Delta_i^* \subset \Delta_i (i = 0, 1..., n-1)$$

such that:

$$f(\Delta_i^*) = \Delta_{i+1}^* \subset \Delta_{i+1} (i = 0, 1, ..., n-2)$$

and $f(\Delta_{n-1}^*) = \Delta_0$. Thus $f^n(\Delta_0^*) = \Delta_0 \supset \Delta_0^*$.

Then, there exists a point x_0 in Δ_0^* such that $f^n(x_0) = x_0$. Since, $f(\Delta_i^*) = \Delta_{i+1}^*$, one has:

$$f^i(x_0) \in \Delta_i^* (i = 0, 1, ..., n-1)$$

Therefore, $x_0 \in \Delta_0$ such that $f_n(x_0) = x_0$ and $f^i(x_0) \in \Delta_i$ ($\forall I \in \{1, ..., n-1\}$).

Corollary 1: If $f([a, b]) \supset [a, b]$ or $f([a, b]) \subset [a, b]$, then f has a fixed point.

Theorem 2: For each positive integer n, f has a periodic point with period n if and only if f has a periodic point with period 3.

Proof: The necessity is manifested.

Now, let $\{x_0, x_1, x_2\}$ is a periodic orbit with period 3 and $x_0 \prec x_1 \prec x_2$. Suppose $f(x_1) = x_0$, then $f(x_0) = x_2$, $f(x_2) = x_1$.

Put $\tilde{\Delta}_0 = [x_0, x_1]$, $\tilde{\Delta}_1 = [x_1, x_2]$ for $\forall n \in \mathbb{N}$: $n \neq 3$.

Let:

$$\Delta_0 = \Delta_1 = ... = \Delta_{n-2} = \tilde{\Delta}_0, \Delta_{n-1} = \tilde{\Delta}_1$$

then:

$$f(\Delta_k) \supseteq \Delta_{k+1} (k = 0, 1, ..., n-2)$$

$$f(\Delta_{n-1}) \supseteq \Delta_0$$

By Theorem 1, there exists a point x^* in Δ_0 such that $f^n(x^*) = x^*$, $f^k(x^*) \in \Delta_k = \tilde{\Delta}_0 (k = 0, 1, ..., n-2)$ and $f^{n-1}(x^*) \in \Delta_{n-1}$.

It can be claimed that x^* is a periodic point with period n. Otherwise there exists an integer $I \in \{0, 1, ..., n-2\}$ such that $f^i(x^*) = f^{n-1}(x^*)$. Then $f^{n-1}(x^*) \in \tilde{\Delta}_0$. That is:

$$f^{n-1}(x^*) \in \tilde{\Delta}_0 \cap \tilde{\Delta}_1 = \{x_1\}$$

So:

$$x^* = f^n(x^*) = f(x_1) = x_0$$

Hence $f^n(x_0) = f^n(x^*) = x^* = x_0$.

Since x_0 is a periodic point with period 3, one has $3|n$. And because $n \neq 3$, then $n \geq 6$. Thus:

$$x_2 = f(x_0) = f(x^*) \in \Delta_1 = \tilde{\Delta}_0 = [x_0, x_1]$$

This contradicts to $x_2 \notin [x_0, x_1]$.

Therefore, f has a periodic point with period n for each positive integer n.

According to Theorem 1 and Theorem 2, the following Corollary 2 is immediately checked.

Corollary 2: If there exist two disjoint closed subintervals $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on X such that $f(\tilde{\Delta}_1), \supseteq \tilde{\Delta}_2$, $f(\tilde{\Delta}_2) \supseteq \tilde{\Delta}_1$ $f(\tilde{\Delta}_1) \supseteq \tilde{\Delta}_1$, then for each integer n, f has a periodic point with period n.

THE HORSESHOES AND PERIODIC POINTS

Let $L \subset X$ a closed interval and $f: L \rightarrow L$ a continuous map.

Theorem 3: If f has a horseshoe then it admits periodic points of all periods.

Proof Let $(\tilde{\Delta}_1, \tilde{\Delta}_2)$ be a horseshoe for f.

Case 1: If $\tilde{\Delta}_1$ and $\tilde{\Delta}_1$ are disjoint. Let $n\geq 1$. Applying Theorem 1 to the chain of intervals $(\Delta_0, \Delta_1, ..., \Delta_n)$ with $\Delta_i = \tilde{\Delta}_2$ for $1 \leq i \leq n-1$ and $\Delta_0 = \Delta_n = \tilde{\Delta}_1$, there exists a point x in $\tilde{\Delta}_1$ such that $f^n(x)$ and $f^k(x) \in \tilde{\Delta}_2$ for $1 \leq k \leq n-1$. Thus, the period of x is exactly n because $\tilde{\Delta}_1$ and $\tilde{\Delta}_1$ are disjoint.

Case 2: $\tilde{\Delta}_1$ and $\tilde{\Delta}_1$ have a common endpoint b. Write $\tilde{\Delta}_1 = [a, b]$ and $\tilde{\Delta}_2 = [b, c]$.

If b is a fixed point, put:

$$d = \text{Ah}[x: b \prec = x, f(x) \in \{a,c\}]$$

Obviously, $b \prec d$ then:

$$\alpha \notin f([b,d]) \text{ and } c \notin f([b,d])$$

Since $(\tilde{\Delta}_1 = \tilde{\Delta}_2)$ be a horseshoe for f, then $[a,c] \subset f([b,c])$, so $\{a,c\} \subset f([d,c])$. Put $\tilde{\Delta}_2^* = [d,c]$, one has $(\tilde{\Delta}_1, \tilde{\Delta}_2)$ is a horseshoe composed of disjoint intervals and the first point of the proof applies.

If b is not a fixed point, applying Theorem 1 to the chain of intervals $(\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_2, \tilde{\Delta}_1$ x, one can takes a point x in $\tilde{\Delta}_1$ such that $f^3(x) = x$ with $f(x) \in \tilde{\Delta}_2$ and $f^2(x) \in \tilde{\Delta}_2$. Let p is the period of x, then p|3. Thus, p = 1 or p = 3. If p = 1 then $x \in \tilde{\Delta}_1 \cap \tilde{\Delta}_2 = \{b\}$. This is impossible. Therefore, p = 3. By theorem 2, f has periodic points of all periods.

Theorem 4: Let $x \in L$, write $x_n = f^n(x)$ for all $n \geq 0$. If f has no horseshoe and $x_n \prec = x_{n+1}$, $x_{m+1} \prec = x_m$, then $x_n \prec = x_m$ and there exists a fixed point $x \in (x_n, x_m)$.

Proof: Suppose $x_m \prec x_n$ and assume that m>n, the case m<n being symmetric. First, one has $f([x_m, x_n]) \cap [x_m, x_n]$. By Corollary 1, there exists a point y in $[x_m, x_n]$ such that f(y) = y. Let $z \in [x_m, x_n]$ is the largest fixed point of f, then $y \prec f(y) (\forall y \in (z, x_n])$. And $x_n \neq z$ (Otherwise $x_n = f^{m-n}(x_n = x_m)$).

Put $k = \min \{i; x_i \prec = z, n+1 \leq i \leq m\}$.

Inductively, $x_i \prec x_{k-1}$ for all integer i from n to k-1. Since, $f(x_{k-1}) = x_k \prec = z \prec x_{k-1}$ (by the definition of k) then $x_{k-1} \notin (z, x_n]$. One has $x_n \prec x_{k-1}$. Suppose $x_i \prec x_{k-1}$ (n<i<k-1), then $x_{i+1} \prec x_{k-1}$. Otherwise $[z, x_i]$, $[x_i, x_k]$ form a horseshoe of f which is a contradiction. For i = k-1, this gives $x_{k-1} \prec x_{k-1}$ which is absurd.

So one has $x_n \prec = x_m$ and $f([x_n, x_m]) \subset [x_n, x_m]$. It can be concluded that there exists a point z in $[x_n, x_m]$ such that f(z) = z.

Theorem 5: Let $x \in L$, p an odd integer. If $f^p(x) \prec = x \prec f(x)$, then f has a periodic point of odd period q with $1 < q \geq p$.

Proof: If $f^p(x) = x$ then x is a periodic point of odd period q = p. And because $f(x) \neq x$, then q>1. Thus it can be assumed that $f^p(x) \prec x$.

If f has horseshoes, by theorem 3, f has periodic points of all periods. One must only assume that f has no horseshoe.

Write $x_n = f^n(x)$ for all $n \geq 0$. Since, $f^p(x) \prec = x \prec f(x)$, then there exist {j: j∈□, 0<j<p} such that $x_{j+1} \prec x_j$.

Let:

$$a = \text{End}\{x_n: x_n \prec = x_{n+1}, 0 \leq n \leq p\}$$

$$b = \text{Ah}\{x_n: x_{n+1} \prec = x_n, 0 \leq n \leq p\}$$

Then $b \prec = f(a)$, $f(b) \prec = a$ and $a \prec = b$ (by theorem 4). Thus $f([a, b]) \supset [a, b]$. So there exists a fixed point $z \in [a, b]$. If there exist $1 \leq i \leq p$ such that $x_i = z$ then:

$$x_0 \prec = a \prec = z = x_p,$$

which is a contradiction. Hence, $x_i \neq z$ $(0 \leq i \leq p)$ and $a \prec z \prec b$.

One can claims that there exist $0 \leq k \leq p$ such that $x_k \prec = x_{k+1} \prec z$ or $z \prec x_{k+1} \prec = x_k$.

Otherwise one gets from place to place that $x_0 \prec z \prec x_1$, $x_2 \prec z \prec x_1$, $x_2 \prec z \prec x_3$,..., $x_{p-1} \prec z \prec x_p$ (p is odd) which is impossible because $x_p \prec x_0 \prec z$.

Suppose for example that $x_k \prec x_{k+1} \prec z$. Define $\Delta_k = [x_k, a]$ and $\Delta_i = [x_i, z]$ for all $0 \leq i \leq p$, $1 \neq k$. Then:

$$f(\Delta_k) \supset [x_{k+1}, b] \supset [x_{k+1}, z]$$

and:

$$f(\Delta_i) \supset [x_{i+1}, z] \text{ for, } 0 \leq i \leq p, i \neq k$$

Thus $(\Delta_0 ...., \Delta_p)$ is a chain of intervals. And because $\Delta_0 \subset \Delta_p$, by Theorem 1, there exists a point $y \in \Delta_0$ such that $f^p(y) = y$. Let q be the period of y then q divides p. Thus q is odd and $q \leq p$. If q = 1 then $y \in \Delta_0 \cap \Delta_j = [x_0, z] \cap [z, x]$and thus y = z. But one also has $y \prec z$ because $y \in \Delta_k = [x_k, a]$. A contradiction. Hence q>1.

To prove Theorem 6, the order relation of odd periodic orbits (Lu and Zhu, 2010) is given first.

Lemma 2: Let X be a CDOLOTS and f: X→X a continuous map. If f has a periodic orbit with period 2n+1 but has no periodic orbit with period 2n-1. Let x_0 be the median point of the orbit. Then one of the following orders holds:

- (i) $x_{2n} \prec x_{2n-2} \prec \ldots \prec x_2 \prec x_0 \prec x_1 \prec x_3 \prec \ldots \prec x_{2n-1}$
- (ii) $x_{2n-1} \prec x_{2n-3} \prec \ldots \prec x_3 \prec x_1 \prec x_0 \prec x_2 \prec$

Theorem 6: If f has a periodic point of odd period different from 1, then there exist two disjoint closed intervals Δ_1 and Δ_2 that do not contain the endpoints of and such that (Δ_1, Δ_2) is a horseshoe for f^2.

Proof: Put:

$$p = \min \{\text{is an odd integer, } n>1, \; f \text{ has a periodic point of period } n\},$$

and let x be a periodic point of period p. According to the order relation of odd periodic orbits, the points of the orbit of x can be ordered in the following order:

$$x_{p-1} \prec x_{p-3} \prec \ldots \prec x_2 \prec x_0 \prec x_1 \prec x_3 \prec \ldots \prec x_{2p-2}$$

or in the reverse order. Suppose the order above holds.

Since, $f([x_0, x_1]) \supset [x_2, x_0]$, then there exists a point d with $x_0 \prec d \prec x_1$ such that f(d) = x0. This implies $d \prec f^2(d) = x_1$.

And because $f^2([x_{p-1}, x_{p-3}]) \supset [x_{p-1}, x_1]$, there exists a point $a \in (x_{p-1}, x_{p-3})$ such that $d \prec f^2(a)$. One has $f^2([a, x_{p-3}]) \supset [x_{p-1}, d]$, thus there exists a point $b \in (a, x_{p-3})$ such that $f^2(b) \prec a$. In the same way, since $f^2([x_{p-3}, d]) \supset [x_{p-1}, d]$, one can choose a point c in (x_{p-3}, d) such that $\bar{f}(c) \prec a$.

Let $\Delta_1 = [a, b]$ and $\Delta_2 = [c, d]$, then it can be easily checked that:

$$f^2(\Delta_1) \cap f^2(\Delta_2) \supset \Delta_1 \cup \Delta_2$$

Since, $x_{p-1} \prec a$ and $d \prec x_1$, then Δ_1 and Δ_2 do not meet the endpoints of L.

Remark: The results in this paper are correct for continuous maps of compact intervals. This study makes it clear that topological structure of spaces have no impact on some properties of periodic points. These properties of periodic points are only related to the order relations of spaces.

CONCLUSION

The following results are obtained:

- The relation between the existence of periodic points and the horseshoes
- Sufficient conditions of the existence for fixed points and periodic points of odd period

ACKNOWLEDGMENT

This work was financially supported by the Scientific Research Fund of Sichuan Provincial Education Department (12ZA098) and Artificial Intelligence of Key Laboratory of Sichuan Province (2012RYY04).

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