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A Combinatorial Proof for Identities on an Odd Number and an Even Number of Right Leaves in p-ary Trees

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Abstract: A binary tree is a plane tree such that each internal vertex has two children. The structure of binary tree is widely used in computer science. The structure of p-ary tree is a generalization of the structure of binary tree such that each internal vertex has p children. This study investigates the number of right leaves in p-ary trees. A leaf is referred as a right leaf if it is the rightmost child of some internal vertex. Moreover, based on p-ary trees, a combinatorial proof of identities related to an odd number and an even number of right leaves in p-ary trees is presented. This study provides an efficient structure for computer and information science and further extensive application of the structure p-ary trees with their right leaves will be investigated in the future.

Key words: Plane tree, binary tree, p-ary tree, leaf

INTRODUCTION

The binary tree is an important structure in computer science. The p-ary tree is a generalization of the binary tree. The p-ary trees with an odd number and an even number of right leaves are studied in the following content. Precisely, a combinatorial proof of two identities related to an odd number and an even number of right leaves in p-ary trees is established.

A p-ary tree with $p\in N^+$ is a plane tree in which each internal vertex has p children. The set of p-ary trees with n internal vertices is counted by the generalized Catalan number (Stanley, 1999):

$$c_p(n)=\frac{1}{(p-1)n+1}\binom{pn}{n}$$

It is also known as order-p Fuss-Catalan number (Fomin and Reading, 2005). By convention, a 2-ary tree is called a complete binary tree and the Catalan sequence $c_2(n)$ is denoted by c(n).

Eu *et al*. (2004), consider and the number of leaves in plane trees. Denoting $P_e(n)$ (resp. $P_o(n)$ the number of plane trees with n edges and an even (resp. odd) number of leaves, Eu *et al*. (2004) obtained the following result by using generating function.

Theorem 1: (Eu *et al*., 2004):

$$P_e(2n)-P_o(2n)=0 \tag{1}$$

$$P_e(2n)-P_o(2n)=0 \tag{2}$$

The above identity is also obtained independently by Bonin *et al*. (1993), Coker (2003) and Klazar (2003) in different ways. Chen *et al*. (2006) presented three parity involutions to describe the identity in a nice combinatorial way.

It is well known that there exists a one-to-one correspondence between the plane trees with n edges and complete binary trees with n internal vertices. This implies that a similar relation as Theorem 1 must hold for complete binary trees and motivates us to try to find a generalization of Theorem 1 for p-ary trees. This generalization is presented by counting some special leaves in a class of restricted p-ary trees which is called compatible p-ary trees.

AN INVOLUTION ON COMPLETE BINARY TREES

First, recall the bijection ϕ between the set P_n of plane trees with n edges and the set CB_n of complete binary trees with n internal vertices. Given a plane tree T with n edges, the following procedure describes the construction of a complete binary tree with n internal vertex:

Step 1: Assume $v_1, v_2, \ldots, v_k$ are the children of the root in T from left to right and T_i is the subtree rooted at v_i for $\le i\le k$, where T_i can be trivial. Construct a plane tree with v_k being the root and v_i is the left child of v_{i+1} for each $1\le 1\le k$-1. If T_i is not trivial, it is the right subtree of v_i, otherwise, v_i has no right child

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Step 2: Repeat Step 1 to each T_i for T_i being nontrivial

Step 3: Add a right leaf, a left leaf, both of right and left leaves to the vertices without right child, left child and leaves respectively in the binary tree obtained from Step 2

The bijection ϕ is exemplified in Fig. 1. From the construction of bijection ϕ, the plane tree with n edges and *k* leaves corresponds to the complete binary tree with *n* internal vertices and k right leaves, where a right leaf is defined as a leaf and it is the right child of some internal vertex. Denoting $CB_e(n)$ (resp. $CB_o(n)$) the set of complete binary trees with n internal vertices and an even (resp. odd) number of right leaves and their cardinality by $B_e(n)$ (resp. $B_o(n)$, the relations (1) and (2) can be expressed equivalently as follows.

Theorem 2: (Eu *et al.*, 2004):

$$B_e(2n) - B_o(2n) = 0$$

$$B_e(2n+1) - B_o(2n+1) = (-1)^{n+1}c(n)$$

This study gives a parity reversing involution on complete binary trees. Notice that for a complete binary tree with n+1 leaves (therefore, it must have n internal vertices), one may change each leaf to internal vertex by attaching it two new leaves (one for left and another for right). After this procedure, a complete binary tree with 2n+1 internal vertices is obtained. Note that for any internal vertex in this tree, either of its two children can be both internal or both leaves. Denote the set of such complete binary trees with n internal vertices by PB_n. It is obvious that:

$$|PB_n| = \begin{cases} 0 & n = 2k \\ c(k) & n = 2k+1 \end{cases}$$

Now the parity reversing involution Ψ on the set $CB_n \backslash PB_n$ is presented. Here, define the parity of a complete binary tree as the parity of the number of its right leaves.

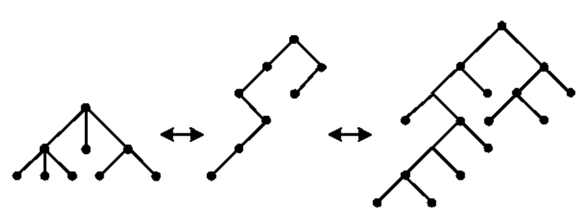

Fig. 1: An example of the bijection ϕ between plane trees and complete binary trees

Assign -1 (resp. 1) to a complete binary tree if the number of its right leaves is odd (resp. even). For any tree T in $CB_n \backslash PB_n$, it must contain at least one internal vertex with one of its children is leaf while the other child is internal which is called a mixed vertex. Traverse T in preorder and suppose v is the first mixed vertex. Interchange the position of two subtrees below v. Let $\Psi(T)$ be the resulting tree and note that if the same procedure is done for $\Psi(T)$, v is still the first mixed vertex encountered. The involution Ψ changes the parity of right leaves in T and $\Psi(T)$, from which another proof of Theorem 2 and also Theorem 1 is derived.

Remark: Theorem 1 can be also verified by applying Cauchy Residue Theorem. Recall that the plane trees with n edges and k leaves obey the Narayana distribution (Dershowitz and Zaks, 1980; Deutsch, 1999), i.e., if denote the number of plane trees of n edges and k leaves by $p_{n,k}$, then:

$$p_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \quad n \geq 1.$$

Therefore,

$$P_e(n) - P_o(n) = \sum_{k=1}^{n}(-1)^k p_{n,k}$$

Substituting $p_{n,k}$ by the Narayana number in the above equation and applying Cauchy Residue Theorem, it gives:

$$\begin{aligned} P_e(n) - P_o(n) &= \sum_{k=0}^{n}(-1)^k \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \\ &= \frac{1}{n}\sum_{k=0}^{n}(-1)^k \mathrm{res}_w (1+w)^n w^{-k-1} \mathrm{res}_u (1+u)^n u^{-k} \\ &= \frac{1}{n}\mathrm{res}_u (1+u)^n \left(\sum_{k=0}^{n}\left(-\frac{1}{u}\right)^k \mathrm{res}_w (1+w)^n w^{-k-1}\right) \\ &= \frac{1}{n}\mathrm{res}_u (1+u)^n \left(1-\frac{1}{u}\right)^n \\ &= (-1)^{\frac{n+1}{2}} \frac{1}{n}\binom{n}{\frac{n-1}{2}} \end{aligned}$$

which implies:

$$P_e(n) - P_o(n) = \begin{cases} 0 & n = 2k \\ (-1)^{k+1}c(k) & n = 2k+1. \end{cases}$$

A GENERALIZATION OF THEOREM 2 IN p-ary TREES WITH ≥3

In fact, there exists a generalization of the above result for p-ary trees. First, some related definitions are

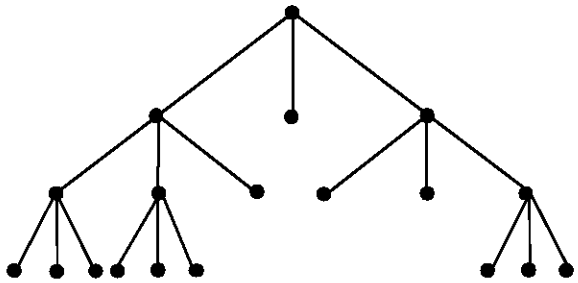

Fig. 2: An example of a compatible 3-ary tree

introduced as preparation. In a p-ary tree T, a vertex u is an elder brother of vertex v if u, v have the same father vertex and u is to the left of v. The youngest child for a vertex v means the rightmost child of v which is also called a right leaf of T if it is a leaf. An internal vertex *v* is called pure if its children are all internal vertices or all leaves, otherwise it is called mixed. A compatible p-ary tree is a p-ary tree satisfying for each internal vertex v, if it is not the youngest child of its father vertex, its nearest elder brother must also be internal. Note that all of v's elder brothers will be internal implied by the definition of compatible p-ary tree.

A compatible 3-ary tree is presented in Fig. 2. Now the generalization of Theorem 2 is given. Denote the set of compatible p-ary trees with n internal vertices by $CT_n^{(p)}$ and denote the set of compatible p-ary trees with n internal vertices and an even (resp. odd) number of right leaves by $CT_e^{(p)}(n)$ (resp. $CT_o^{(p)}(n)$), whose cardinality is $t_e^{(p)}(n)$ (resp. $t_o^{(p)}(n)$).

Theorem 3: The following relations hold:

$$t_e^{(p)}(pn+1)-t_o^{(p)}(pn+1) = (-1)^{(p-1)n+1}C_p(n)$$

$$t_e^{(p)}(n)-t_o^{(p)}(n) = 0, \text{ for } n \neq pk+1$$

Where:

$$c_p(n) = \frac{1}{(p-1)n+1}\binom{pn}{n}$$

is order-p Fuss-Catalan number.

Proof: Observe that for each p-ary tree T with (p-1)n+1 leaves (therefore, T must contain n internal vertices), one can change each leaf into an internal vertex by attaching it p leaves as its children. Then a p-ary tree with pn+1 internal vertices is derived. Note that for any internal vertex v in this tree, either of its children must be all internal or all leaves. Conversely, it is easy to check that

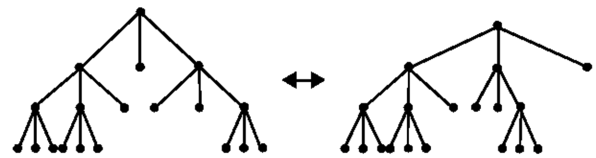

Fig. 3: An example of the involution Φ on the difference set between $CT_n^{(p)}$ and $PT_n^{(p)}$

for any p-ary tree with pn+1 internal vertices satisfying the above requirement, a unique ordinary p-ary tree with n internal vertices can be obtained by removing all the leaves. Denote $PT_n^{(p)}$ the set of p-ary trees with n internal vertices such that for any internal vertex, either of its children are all internal or all leaves. Then:

$$G(x, t) = 1+x(t+G-1)((G-1)^2 + (G-1)+1)$$

The involution ϕ is based on the set $CT_n^{(p)} \backslash PT_n^{(p)}$. Here, define the parity of a p-ary tree as the parity of the number of its right leaves. Assign-1 (resp. 1) to a p-ary tree if the number of its right leaves is odd (resp. even). Given any T in $CT_n^{(p)} \backslash PT_n^{(p)}$, it must contain at least one mixed internal vertex. Traversing T in preorder, assume v is the first mixed vertex encountered. There are two cases according to the youngest child, say w, of v. (1) w is a leaf (2) w is internal.

For Case (1), let u be the nearest elder brother of w which is internal and T_u be the subtree of T rooted at u. Then $\phi(T)$ can be obtained by removing T_u from u and attaching it onto w. For Case (2), let u be the farthest elder brother of w which is a leaf and T_w be the subtree of T rooted at w. Then $\Phi(T)$ can be obtained by removing T_w from w and attaching it onto u.

The compatible 3-ary tree in Fig. 2 and its corresponding compatible 3-ary tree by applying the involution Φ is shown in Fig. 3. Note that the relation for complete binary trees discussed in Section 2 is a special case in Section 3 but it is presented separately to stress the motivation of this study.

Remark: Usually, a compatible 3-ary tree is called a compatible ternary tree. Let G(x, t) be the ordinary generating function of compatible ternary trees, where x and t mark the number of internal vertices and right leaves, respectively. It follows that:

$$G(x, t) = 1+x(t+G-1)((G-1)^2+(G-1)+1)$$

which, substituting (G - 1) by T , implies:

$$T = x(t+T)(T^2+I+1)$$

Table 1: The number of compatible ternary trees $s_{n,k}$ with n internal vertices and k right leaves for n, k from 1 to 7

n \ k	1	2	3	4	5	6	7
$S_{n,k}$							
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	3	2	0	0	0	0
4	1	6	10	4	0	0	0
5	1	10	30	30	9	0	0
6	1	5	70	125	90	21	0
7	1	21	140	385	483	266	51

Applying Lagrange inversion formula to the above equality, it gives:

$$[x^n t^k]T = \frac{1}{n}[x^{n-1}t^k](x+t)^n(x^2+x+1)^n$$

$$= \frac{1}{n}\binom{n}{k}\sum_{\substack{a_1+a_2+a_3=n \\ 2a_1+a_2=k-1}}\binom{n}{a_1, a_2, a_3}$$

$$= \frac{1}{n}\binom{n}{k}\sum_{i=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{n}{k-i-1}\binom{k-i-1}{i}$$

Let $s_{n,k}$ denote the number of compatible ternary trees with n internal vertices and k right leaves. The first few values of $s_{n,k}$ are given in Table 1.

Observe that the diagonal elements in Table 1 equal the n-th Motzkin number M_n. This gives a new combinatorial interpretation of Motzkin sequence. In fact, there is an easy bijection π between compatible ternary trees with (n+1) internal vertices and (n+1) right leaves and Motzkin paths of length n, i.e., lattice paths in the first quadrant from (0, 0) to (n, 0) with steps u = (1, 1), d = (1,-1) and h = (1, 0). Explicitly, given such a compatible ternary tree T with root r, suppose T_1 and T_2 is the first and second subtree of *r* respectively. There are three cases. (1) If both of T_1 and T_2 are trivial, $\pi(T) = \phi$. (2) If T_2 is trivial while T_1 is not trivial, $\pi(T) = h\pi(T_1)$. (3) If neither T_1 nor T_2 is trivial, $\pi(T) = u\pi(T_1)d\pi(T_2)$. Consequently, a Motzkin path is constructed recursively by applying the procedure to $\pi(T_1)$ and $\pi(T_2)$. It is obvious that π is reversible which implies that π is indeed a bijection.

CONCLUSION

This study presents the structures of odd or even right leaves in p-ary trees. Additionally, a different combinatorial proof of two identities by using the structure of p-ary trees is presented. This implies that such structures may be employed as efficient data structures for computer and information science. In the future, the extensive applications of p-ary trees will be investigated.

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