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Preserving Mean-square Stability in the Simulation of Stochastic Differential Delay Equations with Markovian Switching

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Abstract: Stability of stochastic systems with Markovian switching has come to play an important role in information science and engineering. The aim of the study is to discuss the stability of the semi-implicit Milstein scheme of stochastic differential delay equations with Markovian switching. The conditions of the General Mean-square (GMS) stability and Mean-square (MS) stability of the semi-implicit Milstein scheme are given by means of the conditions of the analytical solution. The obtained result shows that the numerical scheme reproduces the stability of the analytical solution to stochastic differential delay equations with Markovian switching under some conditions.

Key words: Markov chain, semi-implicit milstein scheme, mean-square stable, general mean-square stable

INTRODUCTION

Hybrid systems have come to play an important role in information science, engineering and mechanics (Mariton, 1990; Huang *et al.*, 2007; Lou and Cui, 2009; Zhu *et al.*, 2010). One of the important classes of the hybrid systems is the stochastic differential delay equations with Markovian switching (SDDEsMS):

$$dx(t) = f(t, x(t), x(t-\tau), r(t))dt+g(t, x(t), x(t-\tau), r(t))dw(t) \quad (1)$$

where, $r(t)$, $t\geq 0$ be a right-continuous Markov chain on the probability space.

In general, explicit solutions can hardly be obtained for system (1). Thus, it is necessary to develop appropriate numerical methods and to study the properties of these approximate schemes. Stability of numerical Schemes for Stochastic Differential Delay Equations (SDDEs) is essential to avoid a possible explosion of numerical solutions. The convergence and stability properties of the numerical methods for the stochastic ordinary differential equations have been studied by many authors (Mao, 2007; Higham *et al.*, 2002; Hu and Huang, 2011; Zhou and Wu, 2009; Cao *et al.*, 2004; Wang and Zhang, 2006). Mao and Yuan discussed systematically the existence and stability of solutions for stochastic differential equations with Markovian switching (Mao and Yuan, 2006). Rathinasamy and Balachandran (2008) studied the convergence and stability of the semi-implicit Euler-Maruyama method to linear SDDEsMS. Jiang *et al.* (2011) gave the conditions of stability of analytical solutions and the split-step backward Euler method to linear delay stochastic integro-differential equations with Markovian switching. In this study, the linear stochastic differential delay equations with Markovian switching is studied. The main aim of the study is to extend to SDDEsMS and study the General Mean-square (GMS) stability and Mean-square (MS) stability of the semi-implicit Milstein numerical approximations.

STABILITY OF ANALYTICAL SOLUTIONS

Throughout this study, let $(\Omega, F, \{t\}t\geq)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$. Moreover, $|.|$ is the Euclidean norm in R^m and $|\xi|$ is defined by $\|\xi\|=\sup_{-\tau\leq t\leq 0}|\xi(t)|$. Let $\xi(t)$, $t\in[-\tau, 0$ be F_0 measurable and right-continuous and $E\|\xi\|^2<\infty$. Let $w(t)$ be a one-dimensional Brownian motion defined on the probability space. Let $w(t)$, $r(t)$, $t\geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2,\cdots, N\}$ with the generator $\Gamma = (\gamma_{ij})N\times N$ given by:

$$P\{r(t+\delta) = j|r(t)\ i\}\begin{cases}\gamma_{ij}\delta+o(\delta), i\neq j,\\ 1+\gamma_{ij}\delta+o(\delta), i\neq j\end{cases}$$

where, $\delta>0$. Here $\gamma_{ij}\geq 0$ is the transition rate from i to j if $i\neq j$ whil e $\gamma_{ii}=-\sum_{i\neq j}\gamma_{ij}$. The Markov chain $r(t)$ is independent of

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the Brownian motion w(t). It is well known that almost every sample path of r(.) is a right-continuous step function with finite number of simple jumps in any finite subinterval of $R_+ = [0, +\infty)$.

To analyze the Euler-Maruyama scheme as well as to simulate the approximate solution, the following lemma is useful (Mao and Yuan, 2006).

Lemma 1: Given $\Delta>0$, let r^{Δ}_k for $k\geq 0$. Then $\{r^{\Delta}_k, k = 1, 2,\cdots\}$ is a discrete Markov chain with the one-step transition probability matrix:

$$P(\Delta) = (P_{ij}(\Delta))N\times N = e^{\Delta\Gamma}$$

In this study, consider the scalar test equation with Markovian switching

$$dx(t) = [a(r(t))c(t)+b(r(t))x(t-\tau)dt+ [c(r(t))x(t)+d(r(t))x(t-\tau)dw(t) \quad (2)$$

With initial data $x_0 = \xi\in C([-\tau, 0]; R)$ and $r(0) = r_0\in S$, where a(.), b(.), c(.), d(.)$\in$R, w(t) is a standard one-dimensional Brownian motion. The initial data ξ and i_0 could be random, but the Markov property ensures that it is sufficient to consider only the case when both x_0 and i_0 are constants. It is known that the existence and uniqueness of the solutions are ensured under the local Lipschitz condition and the linear growth condition. From Mao and Yuan (2006), the following theorem is obvious.

Theorem 1: If for any $i\in S$, the following inequality:

$$a(i) < -|b(i)| - \frac{1}{2}(|c(i)| + |d(i)|)^2 \quad (3)$$

holds. Then the solution of Eq. 2 is mean-square stable, that is:

$$\lim_{t\to\infty} E|x(t)|^2 = 0 \quad (4)$$

SEMI-IMPLICIT MILSTEIN SCHEME

Now the adaptation of the semi-implicit Milstein method to Eq. 2 leads to a numerical scheme of the following form:

$$y_{n+1} = y_n+\alpha[a(r^{\Delta}_n)y_{n+1-m}]\Delta +(1-\alpha)[a((r^{\Delta}_n)y_n+b(r^{\Delta}_n)yn-m]\Delta +[c(r^{\Delta}_n)y_n+d((r^{\Delta}_n)y_{n-m}]\Delta w_n +c(r^{\Delta}_n)y_n+d(r^{\Delta}_n)y_{n-m}]I_1 +d(r^{\Delta}_n)[c(r^{\Delta}_n)yn-m+d(r^{\Delta}_n)y_{n-2m}]I_2 \quad (5)$$

where $0\leq\alpha\leq 1$, $\Delta>0$ is a stepsize which satisfies $\tau = M\Delta$ for some positive integer m and $t_n = n\Delta$, $r^{\Delta}_n\in S$. y_n is an approximation to x_n if $t_n\geq 0$ then $y_n = \xi(t_n)$. Moreover, $\Delta w_n = w(t_{n+1})-w(t_n)$ are independent. y_n is F_{tn} measurable at the mesh-point t_n. Let I_1 and I_2 denote the two double integrals defined, respectively, by:

$$I_1 = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dw(t)dw(s) = \frac{(\Delta w_n)^2 - \Delta}{2}$$

$$I_2 = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dw(t-\tau)dw(s)$$

The following lemma (Wang and Zhang, 2006) will be useful to the proof of the main result.

Lemma 2: The double integrals I_1 and I_2 satisfy $EI_1 = EI_2 = E(I_1I_2) = 0$:

$$EI_1^2 = EI_2^2 = \Delta^2/2$$

NUMERICAL STABILITY ANALYSIS

In this section the stability of the semi-implicit Milstein numerical method is given.

Definition 1: Under condition 3, a numerical method is said to be mean-square stable(MS--stable), if there exists a $\Delta_0>0$ such that the numerical solution sequence y_n produced by this numerical scheme satisfies $\lim_{n\to\infty} E|y_n|^2 = 0$, for every stepsize $\Delta\in(0, \Delta_0)$ with $\Delta = \tau/m$, where $\Delta_0>0$ dependents on a(.), b(.), c(.), d(.), m is an integer.

Definition 2: Under condition 3, a numerical method is said to be general mean-square stable (GMS--stable), if any application of the method to problem 2 generates numerical approximations y_n which satisfy $\lim_{n\to\infty} E|y_n|^2 = 0$, for every stepsize $\Delta = \tau/m$ and an integer m.

As follows, the main theorem of this study is give:

Theorem 2: Assume that for any $i\in S$, the inequality (3) holds and:

$$L = \max_i\{[\frac{(c^2(i)+d^2(i))(|c(i)|+|d(i)|)^2}{2} +(|a(i)|+|b(i)|)^2 + 2a(i) + 2|b(i)| +(|c(i)| +|d(i)|)^2]/[2|a(i)|(|a(i)|+|b(i)|)]\}$$

- If $L<0$, then for every $\alpha\in[0, 1]$, the semi-implicit Milstein scheme is GMS-stable
- If $L\geq 0$, then for every $\alpha\in(l, 1]$, the semi-implicit Milstein scheme is GMS-stable; for $\alpha\in[0, L]$, it is MS-stable and Δ, where $\Delta' = \max\{\Delta_1, \Delta_2\}$

$$\Delta' = \max\{\min_i\{\frac{1}{|a(r(i))|}\}, \Delta_2\}$$

and

$$\begin{aligned}\Delta_1 = \min\{&\min_i \frac{1}{|a(i)|}, \min_i\{(-2a(i) - 2|b(i)| \\ &-(|c(i)| + |d(i)|)^2)/[(a(i) + |b(i)|)^2 \\ &+(c^2(i) + d^2(i))(|c(i) + d(i))^2)/2]\}\end{aligned}$$

$$\begin{aligned}\Delta_2 = \min_i\{&(-2a(i) - 2|b(i)| \\ &-(|c(i)| + |d(i)|)^2)/((|a(i)| + |b(i)|)^2 \\ &+(c^2(i) + d^2(i))(|c(i) + d(i))^2)/2)\}.\end{aligned}$$

Proof: To analyze the stability of the semi-implicit Milstein scheme, by Lemma 1, the generation of r_n^Δ occurs before computing y_{n+1}, then r_n^Δ is known. Since $r_n^\Delta \in s$, for any $i \in s$, from (5), then

$$\begin{aligned}(1 - a(i)\alpha\Delta)y_{n+1} &= (1 + (1-\alpha)\Delta a(i) + c(i)\Delta w_n)y_n \\ &+\alpha\Delta b(i)y_{n+1-m} + ((1-\alpha)\Delta b(i) + d(i)\Delta w_n)y_{n-m} \\ &+(c^2(i)y_n + c(i)d(i))y_{n-m})I_1 \\ &+(c(i)d(i)y_{n-m} + d^2(i))y_{n-2m})I_2\end{aligned}$$

Note that $E\Delta w_n = 0, E\Delta w_n^2 = \Delta$ and $2\alpha\beta xy \le |\alpha\beta|(x^2 + y^2)$, $\alpha,\beta \in R$. Let $Y_n = E(|y_n|)^2$. It holds that:

$$(1 - \alpha\Delta a(i))^2 Y_{n+1} \le PY_n + QY_{n-m} + RY_{n+1-m} + GY_{n-2m}$$

Where:

$$\begin{aligned}P = &(1 + (1-\alpha)a(i)\Delta)^2 + c^2(i)\Delta \\ &+\frac{\Delta^2}{2}c^2(i)(c^2(i) + |c(i)d(i)| \\ &+(1-\alpha)\Delta|b(i)||1 + (1-\alpha)\Delta a(i)| \\ &+|c(i)d(i)|\Delta + \alpha\Delta|b(i)||1 + (1-\alpha)\Delta a(i)|,\end{aligned}$$

$$\begin{aligned}Q = &(1-\alpha)^2\Delta^2 b^2(i) + d^2(i)\Delta \\ &+\frac{\Delta^2}{2}c^2(i)(d^2(i) + |c(i)d(i)| + \frac{\Delta^2}{2}d^2(i)(c^2(i) \\ &+|c(i)d(i)| + (1-\alpha)\Delta|b(i)||1 + (1-\alpha)\Delta a(i)| \\ &+|c(i)d(i)|\Delta + \alpha\Delta^2(1-\alpha)b^2(i)\end{aligned}$$

$$\begin{aligned}R = &\alpha^2\Delta^2 b^2(i) + \alpha\Delta|b(i)||1 + (1-\alpha)\Delta a(i)| \\ &+\alpha\Delta^2(1-\alpha)b^2(i)\end{aligned}$$

$$G = \Delta^2 d^2(i)(d^2(i) + |c(i)d(i)|)/2$$

Note that by (3) implies $1-\alpha a(i) \neq 0$ for any $i \in S$, then:

$$\begin{aligned}Y_{n+1} \le &\frac{1}{(1 - \alpha\Delta a(i))^2}(P + Q + R + G) \\ &\times\max\{Y_n, Y_{n-m}, Y_{n+1-m}, Y_{n-2m}\}\end{aligned} \quad (7)$$

By recursive calculation, $Y_n \to 0 (n \to \infty)$ if:

$$P + Q + R + G < (1 - \alpha\Delta a(i))^2$$

which is equivalent to:

$$\begin{aligned}&a^2(i)\Delta + 2a(i)(1 - \alpha\Delta a(i)) + \Delta b^2(i) \\ &+2|b(i)||1 + (1-\alpha)\Delta a(i)| + (|c(i)| + |d(i)|)^2 \\ &+(c^2(i) + d^2(i))(|c(i)| + |d(i)|)^2\Delta/2 < 0\end{aligned}$$

If $\Delta < \min_i\{1/|a(i)|\}$, then:

$$\begin{aligned}&2a^2(i) + 2|b(i)| + (|c(i)| + |d(i)|)^2 \\ &+\big[(c^2(i) + d^2(i))(|c(i)| + |d(i)|)^2/2 + (a(i) \\ &+|b(i)|)^2 - 2a(i)(a(i) + |b(i)|)\alpha\big]\Delta < 0.\end{aligned}$$

Since $|1 + (1-\alpha)\Delta a(i)| \le 1 + (1-\alpha)\Delta|a(i)|$

It is obvious that if :

$$\begin{aligned}&2a^2(i) + 2|b(i)| + (|c(i)| + |d(i)|)^2 \\ &+\Big[\frac{c^2(i) + d^2(i)}{2}(|c(i)| + |d(i)|)^2 + (|a(i)| \\ &+|b(i)|)^2 - 2|a(i)|(|a(i)| + |b(i)|)\alpha\Big]\Delta < 0.\end{aligned}$$

Thus $Y_n \to 0 (n \to \infty)$ From (3), $2a(i) + 2|b(i)| + (|c(i)| + |d(i)|)^2 < 0$, then if $L < \alpha \le 1$, then the semi-implicit Milstein method is GMS-stable, as a consequence, when $L < 0$ and $0 < \alpha \le 1$, the method is GMS-stable and if $0 < \alpha \le 1$, then $\lim_{n\to\infty} E|y_n|^2 = 0, \Delta \in (0, \Delta_0)$, thus the method is MS-stable. This proves the theorem.

CONCLUSION

This study is concerned with stability of the semi-implicit Milstein scheme of stochastic differential delay equations with Markovian switching. The GMS-stability and MS-stability of the semi-implicit Milstein method are proved. The obtained result shows that the numerical scheme reproduces the stability of the analytical solution.

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REFERENCES

Cao, W.R., M.Z. Liu and Z.C. Fan, 2004. MS-stability of the Euler-Maruyama method for stochastic differential delay equations. Applied Math. Comput., 159: 127-135.

Higham, D.J., X. Mao and A.M. Stuart, 2002. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. SIAM J. Numer. Anal., 40: 1041-1063.

Hu, P. and C. Huang, 2011. Stability of stochastic theta-methods for stochastic delay integro-differential equations. Int. J. Comput. Math., 88: 1417-1429.

Huang, H., D.W.C. Ho and Y. Qu, 2007. Robust stability of stochastic delayed additive neural networks with Markovian switching. Neural Networks, 20: 799-809.

Jiang, F., Y. Shen and J. Hu, 2011. Stability of the split-step backward Euler scheme for stochastic delay integro-differential equations with Markovian switching. Commun. Nonlinear Sci. Numer. Simul., 16: 814-821.

Lou, X. and B. Cui, 2009. Stochastic stability analysis for delayed neural networks of neutral type with Markovian jump parameters. Chaos Solitons Fractals, 39: 2188-2197.

Mao, X. and C. Yuan, 2006. Stochastic Differential Equations: With Markovian Switching. Imperial College Press, London, UK., ISBN-13: 9781860947018, Pages: 409.

Mao, X., 2007. Stochastic Differential Equations and Applications. 2nd Edn., Horwood Publishing, Chichester, UK., ISBN-13: 978-1-904275-34-3, Pages: 422.

Mariton, M., 1990. Jump Linear Systems in Automatic Control. Taylor and Francis, New York, USA., ISBN-13: 9780824782009, Pages: 299.

Rathinasamy, A. and K. Balachandran, 2008. Mean square stability of semi-implicit Euler method for linear stochastic differential equations with multiple delays and Markovian switching. Applied Math. Comput., 206: 968-979.

Wang, Z. and C. Zhang, 2006. An analysis of stability of Milstein method for stochastic differential equations with delay. Comput. Math. Appl., 51: 1445-1452.

Zhou, S. and F. Wu, 2009. Convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching. J. Comput. Applied Math., 229: 85-96.

Zhu, S., Y. Shen and L. Liu, 2010. Exponential stability of uncertain stochastic neural networks with Markovian switching. Neural Process. Lett., 32: 293-309.