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EWMA Median Control Chart with Variable Sampling Size

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Abstract: In order to reduce the number of observations to signal when the shifts occurs in the process, this study proposed the Exponentially Weighted Moving Average (EWMA) median control charts with Variable Sampling Size (VSS). Give the distribution function of the median value assume that the distribution of observations from a process is normally distributed. Then using the Markov chain method to establish a model to calculate the average number of observations to signal (ANOS) of median control chart. Finally, compare the ANOS of the new model with that of the conventional fixed sampling size EWMA control chart and VSS EWMA mean control chart. The data showed that, the designed new model can effectively reduce the ANOS when the process is out of control. That is to say, the new model effectively improves the efficiency of the monitoring process, shifts can be discovered in a shorter period of time which make a great significance to the actual production.

Key words: EWMA, median control chart, variable sampling size, ANOS

INTRODUCTION

Statistical process control is an effective method for improving product quality and saving firm's productivity (Baxley, 1995; Chou *et al.*, 2006). The major tool of statistical process control is the control chart. The Exponentially Weighted Moving Average (EWMA) charts were first introduced by Roberts (1959) and have been widely used in statistical process control for monitoring process shifts (Chang and Bai, 2001; Crowder, 1989). When the process shifts are small, they can find promptly (Costa, 1997). To improve the inspection efficiency of control charts, VSS Shewhart control chart was firstly proposed by Prabhu *et al.* (1993, 1994) and Costa (1994) respectively, Reynolds and Arnold (2001), Reynolds (1996), Saccucci and Lucas (1990) proposed the EWMA control charts with Variable Sampling Interval (VSI) and so far, the research achievements about VSS control-chart is not so abundant than VSI control-chart (Ji *et al.*, 2006; Wang, 2002). Although, there is a part of literature focused on VSS $\bar{X}$ control chart, very little work has been done on the median design of the VSS control chart (Zhang, 2000).

DESCRIPTION OF THE VSS EWMA MEDIAN CONTROL CHART

Assume that the distribution of observations X from a process is normally distributed and has a mean of μ and a known variance of σ. Then the i-th sample statistic of EWMA chart is:

$$Z_i = \lambda X_i+(1-\lambda)Z_{i-1},\ 0<\lambda\leq 1 \quad (1)$$

where, λ is the exponential weight constant, Z_0 is the starting value and is often taken to be the process target value and the sequentially recorded observations X_i can either be individually observed values from the process or sample averages obtained from rational subgroups. Here, we took X_i as sample median obtained from rational subgroups, then the i-th sample statistic of EWMA chart is:

$$Z_i = \lambda \tilde{X}_i + (1-\lambda)Z_{i-1},\ 0<\lambda\leq 1 \quad (2)$$

When i is zero, $Z_0 = \mu_0$, When the process is in control, then we have:

$$E(Z) = E(\tilde{X}) \quad (3)$$

$$D(Z) = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]D(\tilde{X}) \quad (4)$$

So:

$$\mu_Z = E(\tilde{X}) = \mu_{\tilde{X}} \quad (5)$$

$$\sigma_Z = \sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}\sigma_{\tilde{X}} \quad (6)$$

where, $D(\tilde{X})$ is the variance of distribution function of the median and σ_Z is the standard deviation.

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Then the upper and lower control limits for the VSS EWMA chart can be written as:

$$UCL = \mu_{\tilde{X}} + r\sigma_Z = \mu_0 + \sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2t}]}\sigma_{\tilde{X}} \quad (7)$$

$$LCL = \mu_{\tilde{X}} - r\sigma_Z = \mu_0 - r\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2t}]}\sigma_{\tilde{X}} \quad (8)$$

Generally we adopt its asymptotic form:

$$UCL = \mu_{\tilde{X}} + r\sigma_Z = \mu_0 + r\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}} \quad (9)$$

$$LCL = \mu_{\tilde{X}} - r\sigma_Z = \mu_0 - r\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}} \quad (10)$$

where, r is the control limit coefficient of the VSS EWMA chart. The upper and lower warning limits for the EWMA chart are:

$$UWL = \mu_{\tilde{X}} + r'\sigma_Z = \mu_0 + r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}} \quad (11)$$

$$LWL = \mu_{\tilde{X}} - r'\sigma_Z = \mu_0 - r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}} \quad (12)$$

where, r' ($0<r'<r$) is the warning limit coefficient of the VSS EWMA chart. Then we can get the two regions of the chart.
Where:

$$I_1 = \left(\mu_{\tilde{X}} - r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}, \mu_{\tilde{X}} + r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}\right) \quad (13)$$

$$I_2 = \left[\left(\mu_{\tilde{X}} - r\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}, \mu_{\tilde{X}} - r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}\right)\right] \cup \left[\left(\mu_{\tilde{X}} + r'\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}, \mu_{\tilde{X}} + r\sqrt{\frac{\lambda}{2-\lambda}}\sigma_{\tilde{X}}\right)\right] \quad (14)$$

When designing the control chart with variable sampling size, we choose two sampling size n_1 and n_2, moreover $n_2>n>n_1$. Divide the control charts into center region and alert region. The I_1 is the center region and the alert region is I_2. If the sample statistic falls in the safe region I_1, then take the next sample using n_1. If the sample statistic falls in the warning region I_2, then take the next sample using n_2. If the sample statistic falls outside the control limits, then give a signal and search for the cause.

DISTRIBUTION FUNCTION OF THE MEDIAN

When $X\sim N(\mu, \sigma^2)$, $n = 2s+1$, we can get the distribution function of $\tilde{X}$:

$$G(x) = P\{\tilde{x} \le x\} = \int_{-\infty}^{x} \frac{(2s+1)!}{(s!)^2}[F(\tilde{x})]^s[1-F(\tilde{x})]^s f(\tilde{x})d\tilde{x}$$

Where:

$$F(\tilde{x}) = \int_{-\infty}^{\tilde{x}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du$$

Let $F(\tilde{x})$, then we have:

$$G(x) = \int_0^{F(x)} \frac{1}{B(s+1,s+1)} t^s(1-t)^s dt = \int_0^{\Phi(\frac{x-\mu}{\sigma})} \frac{1}{B(s+1,s+1)} t^s(1-t)^s dt \quad (15)$$

The integral expressions in the right side is incomplete beta distribution function, let $I_\Phi(s+1, s+1)$ denote it or shorthand for I_Φ which can be solved through MATLAB.

Then we get $\mu_{\tilde{X}} = \mu_0$. Let $C_m = \sigma_{\tilde{X}} / \sigma_{\bar{X}}$, then:

$$\sigma_{\tilde{x}} = C_m \sigma_{\bar{x}} = C_m \frac{\sigma}{\sqrt{n}}$$

and C_m can be calculated in the following steps:

Step 1: Let N_0 denote the number of samples from the beginning of the process until it gives a signal. When $\mu = \mu_0$, n is a known constant, then N_0 obey the geometric distribution of parameter q_0:

$$q_0 = P\{|\tilde{X}_i| > r\sigma_Z\} \quad (16)$$

then $1-q_0$ can be calculated based on the distribution function of $\tilde{x}$

Step 2:

$$\Pr(\tilde{X} < C) = 1 - \frac{q_0}{2} = I_{\Phi(C)}$$

Then C can be calculated

Step 3: For:

$$C = r\frac{C_m}{\sqrt{n}}$$

then:

$$C_m = \sqrt{n}\frac{C}{r}$$

Then we get:

$$UCL = \mu_{\tilde{X}} + r\sigma_Z = \mu_0 + r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma \quad (17)$$

$$LCL = \mu_{\tilde{X}} - r\sigma_Z = \mu_0 - r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma \quad (18)$$

$$UWL = \mu_{\tilde{X}} + r'\sigma_Z = \mu_0 + r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma \quad (19)$$

$$LWL = \mu_{\tilde{X}} - r'\sigma_Z = \mu_0 - r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma \quad (20)$$

$$I_1 = \left(\mu_{\tilde{X}} - r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma, \mu_{\tilde{X}} + r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma\right) \quad (21)$$

$$I_2 = \left[\left(\mu_{\tilde{X}} - r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma, \mu_{\tilde{X}} - r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma\right)\right] \cup \left[\left(\mu_{\tilde{X}} + r'\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma, \mu_{\tilde{X}} + r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma\right)\right] \quad (22)$$

AVERAGE NUMBER OF OBSERVATION (ANOS) AND SAMPLE (ANSS) TO SIGNAL

Denote ANOS as the average time to signal, then $ANOS_0$ is the average time to signal when the process mean is in control, $ANOS_1$ is the ANOS when the process mean is out of control and they can be calculated by the Markov chain method as follow:

- Divide the controlled region into 2m+1 equal intervals width for:

$$g = \frac{2r}{2m+1}\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\sigma$$

Every interval denotes a momentary state of Markov chain and each part is regarded as a instantaneous state of Markov chain. Let (L_j, U_j) denote the j-th (j = 1, 2,..., 2m+1) interval, Then we can get:

$$L_j = \mu_0 - r\sigma_Z + \frac{2r\sigma_Z(j-1)}{2m+1}$$
$$U_j = \mu_0 - r\sigma_Z + \frac{2r\sigma_Z j}{2m+1} \quad (23)$$

Let, c_j denote the middle point of the j-th (j = 1, 2,..., 2m+1) interval, then we have:

$$c_j = \mu_0 - r\sigma_Z + \frac{r\sigma_Z(2j-1)}{2m+1} \quad (24)$$

Let, p_{ij} be the one-step transition probability from state i to state j when the progress is in control, then:

$$\begin{aligned} p_{ij} &= Pr(L_j < Z_t < U_j \mid Z_{t-1} = c_i) \\ &= Pr(L_j < \lambda\tilde{X}_t + (1-\lambda)Z_{t-1} < U_j \mid Z_{t-1} = c_i) \\ &= P(L_j < \lambda\tilde{X}_t + (1-\lambda)c_i < U_j) \\ &= G\left(\left(\frac{2j-(1-\lambda)(2i-1)}{(2m+1)\lambda} - 1\right)r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\right) - \\ &\quad G\left(\left(\frac{2(j-1)-(1-\lambda)(2i-1)}{(2m+1)\lambda} - 1\right)r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\right) \end{aligned} \quad (25)$$

where, $\Phi(x)$ is the cumulative distribution of standard normal distribution.

Similarly, calculate the transition probability from state i to state j when the process is out of control p'_{ij}, $\mu = \mu_0 + a\sigma_0$, then:

$$\begin{aligned} p'_{ij} &= G\left(\left(\frac{2j-(1-\lambda)(2i-1)}{(2m+1)\lambda} - 1\right)r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} - a\right) - \\ &\quad G\left(\left(\frac{2(j-1)-(1-\lambda)(2i-1)}{(2m+1)\lambda} - 1\right)r\frac{C_m}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} - a\right) \end{aligned} \quad (26)$$

Define:

$$P_0 = [P_{ij}]_{(2m+1)\times(2m+1)}\ P_1 = [P'_{ij}]_{(2m+1)\times(2m+1)} \quad (27)$$

$$Q_0 = [q_{ij}]_{(2m+1)\times(2m+1)} = (I-P_0)^{-1}$$
$$Q_0 = [q'_{ij}]_{(2m+1)\times(2m+1)} = (I-P_1)^{-1} \quad (28)$$

Then:

$$ANOS_0 = \sum_{j=1}^{2m+1} q_{m+1,j} b_j \quad (29)$$

In a similar way $ANOS_1$ can be written as:

$$ANOS_1 = \sum_{j=1}^{2m+1} q'_{m+1,j} b_j \quad (30)$$

COMPARISON OF FSS AND VSS EWMA MEDIAN CONTROL CHART

To compare the monitoring efficiency of control chart, we should make them have same ANOS when the process is under control. The ANOS rule was proposed by Costa (1994) and Reynolds (1996) and we can prove that ANOS = $E(n_i)$ANSS, if n_1, n_2,... are independent and identically distributed. The ANSS here has the same meaning with ARL when the progress is nuder control (Costa, 1997).

Let N_0 and N_0 denote respectively the number of samples from beginning to signal when, $\mu = \mu_0$ and $\mu = \mu_0 + a\sigma$. If $\mu = \mu_0$, then N_0 obey the geometric distribution of parameter q_0,

$$q_0 = P\{|\tilde{X}_t| > r\sigma_Z\}$$

Let:

$$p_{01} = P\{|\tilde{X}_t| \le r'\sigma_Z\} \qquad p_{02} = P\{r'\sigma_Z \le |\tilde{X}_t| \le r\sigma_Z\} \tag{31}$$

Then the average sampling size is:

$$\bar{n} = \frac{n_1 p_{02} + n_2 p_{01}}{p_{01} + p_{02}} = \frac{n_1 p_{02} + n_2 p_{01}}{1 - q_0} \tag{32}$$

If $\mu = \mu_0 + a\sigma$, denote:

$$p_{i1} = P\{|\tilde{X}_t| \le r'\sigma_Z\} = P\{|\tilde{X}_t| \le r'\frac{C_m}{\sqrt{n_i}}\sigma\} \quad i = 1, 2 \tag{33}$$

$$p_{i2} = P\{r'\sigma_Z \le |\tilde{X}_t| \le r\sigma_Z\} = P\{r'\frac{C_m}{\sqrt{n_i}}\sigma \le |\tilde{X}_t| < r\frac{C_m}{\sqrt{n_i}}\sigma\} \quad i = 1, 2 \tag{34}$$

For VSS c-chart, select appropriate r' to make the formular established:

$$n = \frac{n_1 p_{02} + n_2 p_{01}}{p_{01} + p_{02}} = \frac{n_1 p_{02} + n_2 p_{01}}{1 - q_0} \tag{35}$$

p_{01}, p_{02} satisfy:

$$p_{01} = \frac{n - n_1}{n_2 - n_1}(1 - q_0), \; p_{02} = \frac{n_2 - n}{n_2 - n_1}(1 - q_0) \tag{36}$$

Then we can decide r' by Eq. 21.

Let λ take different values, select appropriate sampling size n_1 and n_1 to make the two charts have the same ANSS when $\mu = \mu_0$. Then calculate the ANSS of the two control charts when $\mu = \mu_0 + a\sigma_0$, the smaller the ANSS is, the higher the efficiency of the control chart will be.

For VSS c-chart, when $\mu = \mu_0$, $n = 5$, $r = 3$, $n_1 = 3$, $n_2 = 7$, we have $p_{02} = p_{01} = 0.4985$ from Eq. 21. From Eq. 17, we get $\bar{n} = 5$, then we decide r' according to n.

From $G(c') = P\{\tilde{x} \le c'\} = I_{\Phi(c')} = (0.4985/2+0.5) = 0.74925$. We get $\Phi(c') = 0.6401$, $c' = 0.3587$. From r' $cm/\sqrt{n} = 0.3587$, we get $r' = 0.6699$.

In a similar way, we can calculate the average sampling size $\bar{n}$, then we can know ANSS.

COMPARISON OF MEAN AND MEDIAN CONTROL CHART

Then,we compare the ANSS of VSS mean and VSS median control chart. As well,select appropriate sampling size n_1 and n_2 to make the two charts have the same ANSS when $\mu = \mu_0$. For n can only be integer here, so, we can only make both approximately equal.

CONCLUSION

From Table 1, we know that no matter what value the λ is, the ANSS of VSS EWMA control chart is smaller than that of FSS EWMA control chart when a is lager than 0.75. However, when a is lager than 0.75, the efficiency advantage is not so large.

From Table 2, we know that no matter what value the λ and a is, the ANSS of VSS median EWMA control chart is smaller than that of VSS EWMA mean control chart. Moreover, the advantage is much more obvious when the progress has a smaller offset. Just as we imagined, when the sampling result is good enough, we reduce the next sampling sample size, the other way around, we increase the sample size to discover the offset faster. So, the VSS control chart is more suitable than FSS control chart to apply to progress in practice. Besides, conventional control charts are monitoring the mean or variance, this study monitor the median and it has a more efficient result which is significant to the actual production.

Table 1: ANNS of FSS and VSS EWMA control charts

	$\lambda = 0.1$		$\lambda = 0.2$		$\lambda = 0.3$		$\lambda = 0.4$		$\lambda = 0.5$	
a	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
0.00	835.5790	837.4379	550.9741	552.9280	453.9169	455.8554	406.6609	408.5787	380.0603	381.9653
0.50	12.6223	13.7329	12.3031	13.4662	13.5404	14.7998	15.6738	17.0056	18.6412	19.9985
0.75	7.1679	7.0240	6.2076	6.0822	6.1312	6.0518	6.4603	6.4136	7.1198	7.0821
1.00	5.0507	4.5658	4.1490	3.6955	3.8673	3.4236	3.8227	3.3868	3.9339	3.5057
1.25	3.9413	3.3708	3.1557	2.6144	2.8587	2.3235	2.7245	2.1916	2.6804	2.1490
1.50	3.2635	2.6819	2.5815	2.0184	2.3085	1.7486	2.1485	1.5893	2.0512	1.4921
1.75	2.8021	2.2253	2.2218	1.6538	1.9668	1.4000	1.7912	1.2243	1.6738	1.1067
2.00	2.4561	1.8834	1.9887	1.4191	1.7196	1.1502	1.5329	0.9633	1.4164	0.8466

Table 2: ANSS of VSS mean and VSS median control chart

	λ = 0.1		λ = 0.2		λ = 0.3		λ = 0.4		λ = 0.5	
a	Mean	Median	Mean	Median	Mean	Median	Mean	Median	Mean	Median
0.00	841.1630	837.4379	550.6036	552.9282	465.4111	455.8554	421.0311	408.5787	397.4182	381.9653
0.50	37.4473	13.7329	41.5131	13.4662	47.6676	14.7998	55.0226	17.0056	63.4609	19.9985
0.75	19.1275	7.0240	19.7964	6.0822	22.2646	6.0518	25.7417	6.4136	30.1445	7.0821
1.00	11.8096	4.5658	11.3176	3.6955	12.1494	3.4236	13.6641	3.3868	15.7828	3.5057
1.25	8.2990	3.3708	7.4824	2.6144	7.6542	2.3235	8.2917	2.1916	9.3156	2.1490
1.50	6.2976	2.6819	5.4338	2.0184	5.3344	1.7486	5.5656	1.5893	6.0503	1.4921
1.75	5.8028	2.2253	4.8538	1.6538	4.6113	1.4000	4.6531	1.2243	4.8945	1.1067
2.00	4.1595	1.8834	3.4023	1.4191	3.1505	1.1502	3.0900	0.9633	3.1528	0.8466

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