OPEN ACCESS Journal of Applied Sciences

ISSN 1812-5654
DOI: 10.3923/jas.2018.

Research Article
Fixed Point Theorems in Ordered S_b-Metric Spaces by Using (α, β)-Admissible Geraghty Contraction and Applications

[1]Bagathi Srinuvasa Rao, [1]Gajula Naveen Venkata Kishore, [2]Muhammad Sarwar and [1]Nalamalapu Konda Reddy

[1]Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, 522502 Andhra Pradesh, India
[2]Department of Mathematics, University of Malakand, Chakadara Dir (L), Khyber Pakhtunkhwa, Pakistan

Abstract
The purpose of this paper was to prove some fixed point theorems in S_b-metric spaces by using (α, β)-admissible Geraghty type rational contractive conditions and some suitable examples have been provided with relevant to the results. Also, an application to Homotopy theory as well as integral equations were given.

Key words: (α, β)-admissible, geraghty type rational contraction, S_b-metric spaces, completeness, partial ordering and fixed point

Citation: Bagathi Srinuvasa Rao, Gajula Naveen Venkata Kishore, Muhammad Sarwar and Nalamalapu Konda Reddy, 2018. Fixed point theorems in ordered -metric spaces by using (α, β)-admissible geraghty contraction and applications. J. Applied Sci., CC: CC-CC.

Corresponding Author: G.N.V. Kishore, Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, 522502 Andhra Pradesh, India Tel: +918096844888

Copyright: © 2018 Bagathi Srinuvasa Rao *et al.* This is an open access article distributed under the terms of the creative commons attribution License, which permits unrestricted use, distribution and reproduction in any medium, provided the original author and source are credited.

Competing Interest: The authors have declared that no competing interest exists.

Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

In Banch[1] introduced the notion of Banach contraction principle. It is most fundamental tool in nonlinear analysis and some results related with generalization of various type of metric spaces[2-5].

In recent time, Sedghi *et al*.[6] described S_b-metric spaces by applying the concept of S and b-metric spaces and established some fixed point results in S_b-metric spaces. Subsequently to improve many author's established numerous results on -metric spaces[7-10].

In Geraghty[11] studied a generalization of Banach contraction principle. In Samet *et al*.[12] initiated the concept of α-contractive and α-admissible mappings and proved fixed point theorems on complete metric spaces for such class of mappings. In Cho *et al*.[13] initiated the concept of α-Geraghty contractive type mappings. On the other hand, Karapinar[14] established the existence of a unique fixed point for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contractive type mapping. Later on, Chandok[15] illustrated the theory of (α, β)-admissible Geraghty type contractive mappings. Also very recently, Gupta *et al*.[16] proved some fixed point results in ordered metric spaces under the (ψ, β)-admissible Geraghty contractive type mappings.

Subsequently, this type of research has been studied by several investigators[17-25].

The aim of present article was to prove unique fixed point theorems for (α, β)-admissible Geraghty type contraction in ordered S_b-metric spaces. The obtained results generalized, unified and modified some recent theorems in the literature. Some suitable example and an applications to Homotopy theory as well as integral equations were given here to illustrate the usability of the obtained results.

Firstly, recall some definitions, lemmas and examples.

PRELIMINARIES

Definition: ([6]) Let: S_b: $X^3 \to [0, 1)$ be a mapping defined on a non-empty set X and $b \geq 1$ be a given real number. Suppose that the mapping S_b satisfies the following properties:

$$(S_b1)\ 0<S_b(l, m, n) \text{ for all } l, m, n \in X \text{ with } l \neq m \neq n \neq l$$

$$(S_b2)\ S_b(l, m, n) = 0 \Leftrightarrow l = m = n$$

(S_b3) $0<S_b(l, m, n) \leq b\,(S_b(l, l, x)+S_b(m, m, x)+S_b(n, n, x))$ for all $l, m, n \in X$. Then, the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark: ([6]) It must be noted that, the class of S_b-metric spaces is definitely larger than that of S-metric spaces. In fact, each S-metric space is a S_b-metric space whenever b = 1.

Following example shows that a S_b-metric space on X need not be a S-metric spaces.

Example: ([6]) Let (X, S) be S-metric space and $S_*(l, m, n) = S(l, m, n)^k$ where $k>1$ is a real number. Note that (X, S_*) is not necessarily S-metric space but S_* is a S_b-metric with $b = 2^{2(k-1)}$.

Definition: ([6]) Let (X, S_b) be a S_b-metric space. Then, we define the open ball $Bs_b\,(l, r)$ and closed ball $Bs_b\,[l, r]$ with centre $l \in X$ and radius $r>0$ as following respectively:

$$Bs_b\,(l, r) = \{m \in X : S_b(m, m, l) < r\}$$

and:

$$Bs_b[l, r] = \{m \in X\ S_b:(m, m, l) \leq r\}$$

Lemma: ([6]) In a S_b-metric space, we have $S_b(u, u, w) \leq 2b\, S_b\,(u, u, v)+b^2 S_b(v, v, w)$.

Definition: ([6]) If (X, S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

- S_b-Cauchy sequence if, for each $\epsilon>0$, there is an integer $n_0 \in Z^+$ such that $S_b\,(x_n, x_n, x_m)<\epsilon$ for each $n, m \geq n_0$
- S_b-convergent to a point $x \in X$ if, for each $\epsilon>0$, there is an integer $n_0 \in Z^+$ such that $S_b\,(x_n, x_n, x)<\epsilon$ or $S_b\,(x, x, x_n)<\epsilon$ for all $n \geq n_0$ and denoted by $\lim_{n\to\infty} x_n = x$

Definition: ([6]) A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma: ([6]) If (X, S_b) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then:

$$\frac{1}{2b} S_b\,(y, y, x) \leq \lim_{n\to\infty} \inf S_b\,(y, y, x_n) \leq \lim_{n\to\infty} \sup S_b\,(y, y, x_n) \leq 2b S_b\,(y, y, x)$$

and:

$$\frac{1}{b^2} S_b\,(x, x, y) \leq \lim_{n\to\infty} \inf S_b\,(x_n, x_n, y) \leq \lim_{n\to\infty} \sup S_b\,(x_n, x_n, y) \leq b^2 S_b\,(x, x, y)$$

for all $y \in X$. Specifically, if x = y then $\lim_{n\to\infty} S_b\,(x_n, x_n, y) = 0$.

Definition: Let $E: X \to X$ be a self-mapping and $\alpha, \beta: X \times X \times X \to R^+$ defined on non-empty set X. Then, the mapping E is said to be (α, β)-admissible mapping, if $\alpha(x, x, y) \geq 1$ and $\beta(x, x, y) \geq 1$ implies $\alpha(Ex, Ex, Ey) \geq 1$ and $\beta(Ex, Ex, Ey) \geq 1$ for all $x, y \in X$.

Definition: Let (X, S_b) be a S_b-metric space, $\alpha, \beta: X \times X \times X \to [0, \infty]$ be a mappings defined on non-empty set X. We say that X is a (α, β)-regular if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$, $\alpha(x_n, x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_n, x_{n+1}) \geq 1$ and $x_n \precsim x_{n+1}$ then there exist a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that:

$$\alpha\left(x_{n_k}, x_{n_k}, x_{n_k+1}\right) \geq 1$$

and:

$$\beta\left(x_{n_k}, x_{n_k}, x_{n_k+1}\right) \geq 1$$

and $x_{n_k} \precsim x_{n_k+1}$ for all $k \in N$ and $\alpha(x, x, Ex) \geq 1$ and $\beta(x, x, Ex) \geq 1$.

RESULTS AND DISCUSSIONS

Let $\Omega = \{\Omega/\Omega: [0, \infty) \to [0, 1)\}$ be a family of function then $\{t_n\}$ be a any bounded sequence of positive reals such that $(t_n) \to 1$ ast$_n \to 0$. Let $\Phi = \{\Phi: \Phi: [0, \infty) \to [0, \infty)\}$ be a family of functions such that Φ is continuous, strictly increasing and $\Phi(0) = 0$.

Definition: Let (X, S_b) be a S_b-metric space, $\alpha, \beta: X \times X \times X \to R^+$ and $E: X \to X$ is said to be (α, β)-Geraghty type-I and type-II rational contractive mapping if, there exists $\Omega \in \Omega$ such that for all $x, y \in X$, satisfies the following conditions:

$$(\phi\left(4b^5 S_b\left(Ex, Ex, Ey\right)\right) + r)^{\alpha(x, x, Ex)\beta(y, y, Ey)} \leq \Omega((N_E{}^i(x, y)))(N_E{}^i(x, y)) + r \quad (1)$$

$$\alpha(x, x, Ex)\beta(y\, y, Ey)\phi\left(4b^5 S_b\left(Ex, Ex, Ey\right)\right) \leq \Omega((N_E{}^i(x, y)))(N_E{}^i(x, y)) \quad (2)$$

For all $x, y \in X$, x is comparable to y, I = 3 or 4 and $\phi \in \Phi$, $r \geq 1$. Where:

$$N_E{}^4(x, y) = \max\{S_b(x, x, y), S_b(x, x, Ex), S_b(y, y, Ey), \frac{S_b(x, x, Ey)S_b(y, y, Ex)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)}\}$$

$$N_E{}^3(x, y) = \max\{S_b(x, x, y), \frac{S_b\left(x, x, Ex\right)S_b\left(y, y, Ey\right)}{1 + S_b\left(x, x, y\right) + S_b\left(Ex, Ex, Ey\right)}, \frac{S_b\left(x, x, Ey\right)S_b\left(y, y, Ex\right)}{1 + S_b\left(x, x, y\right) + S_b\left(Ex, Ex, Ey\right)}\}$$

Theorem: Let $(X, S_b, \preceq)$ be complete ordered S_b-metric space, $\alpha, \beta: X \times X \times X \to R^+$ and $E: X \to X$ be satisfies:

- E is an (α, β)-admissible mapping
- E is an (α, β)-Geraghty type-I rational contractive mapping with i = 4
- There exist $x_0 \in X$ such that $x_0 \preceq E\, x_0$ with $\alpha(x_0, x_0, Ex_0) \geq 1$ and $\beta(x_0, x_0, Ex_0) \geq 1$ for all $x_0 \neq Ex_0$
- Either E is continuous or X is (α, β)-regular

Then E has a unique fixed point in X.

Proof Let $X_0 \in X$ such that $\alpha(x_0, x_0, Ex_0) \geq 1$ and $\beta(x_0, x_0, Ex_0) \geq 1$, since E is self-map, then $\exists$ a sequence $\{x_n\}$ in X such that $x_{n+1} = E\, x_n$, n = 0, 1, 2, 3.

Case I: If $x_n = Ex_n = x_{n+1}$, then clearly x_n is a fixed point of E.

Case II: Assume $x_n \neq E\, x_n$, $\forall n$.

Since $x_0 \preceq Ex_0 = x_1$ and by definition of E, we have:

$$x_0 \preceq Ex_0 \preceq E^2x_0 \preceq E^3x_0 \ldots\ E^nx_0 \preceq E^{n+1}x_0 \preceq \ldots$$

Since E is (α, β)-admissible mapping, $\alpha(x_0, x_0, Ex_0) = \alpha(x_0, x_0, x_1) \geq 1$:

$$\alpha(Ex_0, Ex_0, Ex_1) = \alpha(x_1, x_1, x_2) \geq 1,\ \alpha(Ex_1, Ex_1, Ex_2) = \alpha(x_2, x_2, x_3) \geq 1$$

Hence by induction, we get $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

Similarly, $\beta(x_n, x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

Now:

$$(4b^5 S_b(Ex_0, Ex_0, E^2x_0)) + r = (4b^5 S_b(Ex_0, Ex_0, Ex_1))) + r$$

$$\leq (\varphi(4b^5 S_b(Ex_0, Ex_0, E^2x_0)) + r)\alpha(x_0, x_0, Ex_0)\ \beta(x_1, x_1, Ex_1)$$

$$\leq \Omega\ ((N^4{}_E(x_0, x_1)))\ (N^4{}_E(x_0, x_1)) + r$$

Where:

$$N_E^4(x_0, x_1) = \max\left\{\begin{matrix} S_b(x_0, x_0, x_1), S_b(x_0, x_0, Ex_0), S_b(x_1, x_1, Ex_1) \\ \dfrac{S_b(x_0, x_0, Ex_1)S_b(x_1, x_1, Ex_0)\}}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2x_0)} \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} S_b(x_0,x_0,Ex_0),S_b(x_0,x_0,Ex_0),S_b(Ex_0,Ex_0,E^2x_0) \\ \dfrac{S_b(x_0,x_0,E^2x_0)S_b(Ex_0,Ex_0,Ex_0)\}}{1+S_b(x_0,x_0,Ex_0)+S_b(Ex_0,Ex_0,E^2x_0)} \\ \end{matrix}\right\}$$

$$= \max\ \{S_b\ (x_0,\ x_0,\ Ex_0),\ S_b\ (Ex_0,\ Ex_0,\ E^2x_0)\}$$

Thus:

$$(4b^5S_b(Ex_0,Ex_0,E^2x_0)) \le \Omega\left(\phi\left(\max\left\{\begin{matrix} S_b(x_0,x_0,Ex_0) \\ S_b(Ex_0,Ex_0,E^2x_0)\end{matrix}\right\}\right)\right)$$

$$\phi\left(\max\left\{\begin{matrix} S_b(x_0,x_0,Ex_0) \\ S_b(Ex_0,Ex_0,E^2x_0)\end{matrix}\right\}\right)$$

Also:

$$(4b^5S_b(E^2x_0,E^2x_0,E^3x_0)) \le \Omega\left(\phi\left(\max\left\{\begin{matrix} S_b(Ex_0,Ex_0,E^2x_0) \\ S_b(E^2x_0,E^2x_0,E^3x_0)\end{matrix}\right\}\right)\right)$$

$$\phi\left(\max\left\{\begin{matrix} S_b(Ex_0,Ex_0,E^2x_0) \\ S_b(E^2x_0,E^2x_0,E^3x_0)\end{matrix}\right\}\right)$$

Continuing this way we can conclude that:

$$(4b^5S_b(E^{n+1}x_0,E^{n+1}x_0,E^{n+2}x_0)) \le \Omega\left(\phi\left(\max\left\{\begin{matrix} S_b(E^nx_0,E^nx_0,E^{n+1}x_0) \\ S_b(E^{n+1}x_0,E^{n+1}x_0,E^{n+2}x_0)\end{matrix}\right\}\right)\right)$$

$$\phi\left(\max\left\{\begin{matrix} S_b(E^nx_0,E^nx_0,E^{n+1}x_0) \\ S_b(E^{n+1}x_0,E^{n+1}x_0,E^{n+2}x_0)\end{matrix}\right\}\right)$$

If S_b (E^nx_0, E^nx_0, $E^{n+1}x_0$)$\le S_b$($E^{n+1}x_0$, $E^{n+1}x_0$, $E^{n+2}x_0$), which is contradiction.

Hence S_b($E^{n+1}x_0$, $E^{n+1}x_0$, $E^{n+2}x_0$)$\le S_b$ (E^nx_0, E^nx_0, $E^{n+1}x_0$). Thus, {S_b (E^nx_0, E^nx_0, $E^{n+1}x_0$)} is non-increasing and must converges to a real number $\xi\ge 0$. Such that $\lim_{n\to\infty} S_b$ (E^nx_0, E^nx_0, $E^{n+1}x_0$) = ξ. If $\xi>0$ which is contradiction. Hence $\xi = 0$. Thus $\lim_{n\to\infty} S_b$ (E^nx_0, E^nx_0, $E^{n+1}x_0$) = 0.

Now we prove that {E^nx_0} is a Cauchy sequence in (X, S_b). On contrary assume that {E^nx_0} is not Cauchy sequence. Then there exist $\in>0$ and monotonically increasing sequence of natural numbers {m_k} and {n_k} such that $n_k>m_k$:

$$S_b(E^{m_k}x_0,E^{m_k}x_0,E^{n_k}x_0) \ge \in \tag{3}$$

and:

$$S_b(E^{m_k}x_0,E^{m_k}x_0,E^{n_{k-1}}x_0) < \in \tag{4}$$

From Eq. 3 and 4, we have:

$$\in \le S_b(E^{m_k}x_0,E^{m_k}x_0,E^{n_k}x_0)$$

$$\le 2b\ S_b(E^{m_k}x_0,E^{m_k}x_0,E^{n_{k+1}}x_0) + b^2\ S_b(E^{m_{k+1}}x_0,E^{m_k+1}x_0,E^{n_k}x_0)$$

Letting $k\to\infty$:

$$\phi(4b^3\in) + r \le \lim_{k\to\infty}\phi(4b^5S_b(E^{m_k+1}x_0,E^{m_k+1}x_0,E^{n_k}x_0)) + r$$

$$\le \lim_{k\to\infty}\left(4b^5S_b\left(Ex_{m_k},Ex_{m_k},Ex_{n_k\text{-}1}\right)+r\right)^{\alpha\left(x_{m_k},\,x_{m_k},\,Ex_{m_k}\right)\beta\left(x_{n_k-1},\,x_{n_k-1},\,Ex_{n_k-1}\right)}$$

$$\le \lim_{k\to\infty}\Omega\left(\phi\left(N^4{}_E(x_{m_k},x_{n_k-1})\right)\right)\phi\left(N^4{}_E(x_{m_k},x_{n_k-1})\right)+r \tag{5}$$

Where:

$$\lim_{k\to\infty}N^4{}_E(x_{m_k},x_{n_k-1})$$

$$= \lim_{k\to\infty}\ \max\left\{\begin{matrix} S_b\left(E^{m_k}x_0,E^{m_k}x_0,\ E^{n_{k-1}}x_0\right),S_b\left(E^{m_k}x_0,E^{m_k}x_0,\ E^{m_{k+1}}x_0\right), \\ S_b\left(E^{n_{k-1}}x_0,E^{m_{k-1}}x_0,\ E^{n_k}x_0\right) \\ \dfrac{S_b\left(E^{m_k}x_0,E^{m_k}x_0,\ E^{n_k}x_0\right)S_b\left(E^{n_{k-1}}x_0,E^{m_{k-1}}x_0,\ E^{m_{k+1}}x_0\right)}{1+S_b\left(E^{m_k}x_0,E^{m_k}x_0,\ E^{n_{k-1}}x_0\right)+S_b\left(E^{m_{k+1}}x_0,E^{m_{k+1}}x_0,\ E^{n_k}x_0\right)}\end{matrix}\right\}$$

But:

$$\lim_{k\to\infty}\left\{\frac{S_b\,(E^{m_k}x_0,\,E^{m_k}x_0,\,E^{n_k}x_0)\,S_b\,(E^{n_{k-1}}{}_{x_0},\,E^{n_{k-1}}{}_{x_0},\,E^{m_{k+1}}{}_{x_0})}{1+S_b\,(E^{m_k}x_0,\,E^{m_k}x_0,\,E^{n_{k-1}}x_0)+S_b\,(E^{m_{k+1}}x_0,\,E^{m_{k+1}}x_0,\,E^{n_k}x_0)}\right\}$$

$$= \lim_{k\to\infty}\left\{\left(\begin{matrix}[2bS_b\,(E^{m_k}x_0,\,E^{m_k}x_0,\,E^{n_{k-1}}x_0)+b^2S_b\,(E^{n_{k-1}}x_0,\,E^{n_{k-1}}x_0,\,E^{m_k}x_0) \\ \dfrac{2bS_b\,(E^{n_{k-1}}x_0,\,E^{n_{k-1}}x_0,\,E^{m_k}x_0)+b^2S_b\,(E^{m_k}x_0,\,E^{m_k}x_0,\,E^{m_k}x_0)}{1+S_b\,(E^{m_k}x_0,\,E^{m_k}x_0,\,E^{n_{k-1}}x_0)+S_b\,(E^{m_{k+1}}x_0,\,E^{m_{k+1}}x_0,\,E^{n_k}x_0)} \\ \end{matrix}\right)\right\} \le 4b^3\in$$

Now from Eq. 5:

$$\phi(4b^3\in) \le \lim_{k\to\infty}\Omega\left(\phi\left(N^4{}_E(x_{m_k},x_{n_k-1})\right)\right)\phi(4b^3\in)$$

It is clear that:

$$\lim_{k\to\infty}\Omega\left(\phi\left(N^4{}_E(x_{m_k}, x_{n_k-1})\right)\right) = 0$$

Hence:

$$\lim_{k\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, 1\, x_{n_k}) = 0$$

Which is contradiction. Hence $\{E^n x_o\}$ is a Cauchy sequence in (X, S_b). Because of completeness of (X, S_b), there is an $\nu \in X$ with $\{E^n x_o\} \to \nu \in (X, S_b)$.

Assume that E is continuous. Therefore:

$$\nu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Ex_n = E \lim_{n\to\infty} x_n = E\,\nu$$

Now, assume that X is $(\alpha$- $\beta)$regular. Therefore, there exists a sub sequence $\{x_{nk}\}$ of $\{x_n\}$ such that:

$$\alpha(x_{n_k\text{-}1}, x_{n_k\text{-}1}, x_{n_k}) \geq 1$$

$$\beta(x_{n_k\text{-}1}, x_{n_k\text{-}1}, x_{n_k}) \geq 1$$

for all $k \in N$ and $\alpha(\nu, \nu, E\,\nu) \geq 1$ and $\beta(\nu, \nu, E\,\nu) \geq 1$. Since $Ex_{n_k} \to \nu$ and (X, S_b) is regular, it follows x_{n_k} is comparable to ν.

Suppose $E\,\nu \neq \nu$. From (1) and by the definition of ϕ, by known Lemma:

$$\phi(2b^2 S_b(E\nu, E\nu, \nu)) + r \leq \lim_{n\to\infty}\inf \phi(4b^5 S_b(E\nu, E\nu, E^{n_k+1}x_0)) + r$$

$$\leq \liminf_{n\to\infty}\left(\phi\left(4b^5 S_b(E\upsilon, E\upsilon, Ex_{n_k})\right) + r\right)^{\alpha(\upsilon, \upsilon, E\upsilon)\beta(x_{n_k}, x_{n_k}, Ex_{n_k})}$$

$$\leq \lim_{k\to\infty}\inf\left(\Omega(\phi(N_E{}^4(\upsilon, x_{n_k})\right)\phi(N_E{}^4\left(\upsilon, x_{n_k}\right) + r \qquad (6)$$

Here:

$$\lim_{k\to\infty} N_E{}^4(\upsilon, x_{n_k}) = \lim_{n\to\infty}\max\left\{\begin{array}{c} S_b(\upsilon, \upsilon, x_{n_k}), S_b(\nu, \upsilon, E\upsilon,), S_b(x_{n_k}, x_{n_k}, Ex_{n_k}) \\ \dfrac{S_b(\upsilon, \upsilon, Ex_{n_k})S_b(x_{n_k}, x_{n_k}, E\upsilon,)}{1 + S_b(\upsilon, \upsilon, x_{n_k}) + S_b(E\upsilon, E\upsilon, Ex_{n_k})} \end{array}\right\}$$

$$= S_b(\nu, \nu, E\,\nu)$$

Hence from Eq. 6:

$$\phi\left(2b^2 S_b(E\upsilon, E\upsilon, \upsilon)\right) \leq \liminf_{k\to\infty}\left(\Omega(\phi(N_E{}^4(\upsilon, x_{n_k})\right)\phi(S_b\left(\nu, \upsilon, E\upsilon,\right))$$

So, we have:

$$\frac{\phi\left(2b^2 S_b(E\upsilon, E\upsilon, \upsilon)\right)}{\phi(S_b\left(\upsilon, \upsilon, E\upsilon,\right))} \leq \lim_{k\to\infty}\inf\left(\Omega(\phi(N_E{}^4(\upsilon, x_{n_k})\right)$$

That is:

$$\lim_{k\to\infty}\inf\left(\Omega(\phi(N_E{}^4(\upsilon, x_{n_k})\right) = 0$$

Consequently:

$$\lim_{k\to\infty} N_E{}^4(\upsilon, x_{n_k}) = 0$$

And hence $S_b(\nu, \nu, E\nu) = 0$ that is $\nu = E\nu$. Therefore, ν is fixed point of E.

Further to prove the uniqueness, suppose that ν^* is also anther fixed point of E such that $\nu \neq \nu^*$ and $\alpha(\nu, \nu, E\,\nu) \geq 1$, $\alpha(\nu^*, \nu^*, E\,\nu^*) \geq 1$ and $\beta(\nu, \nu, E\,\nu) \geq 1$, $\beta(\nu^*, \nu^*, E\,\nu^*) \geq 1$.

Consider:

$$\phi(4b^5 S_b(\upsilon, \upsilon, \upsilon^*)) + r = \phi(4b^5 S_b(E\upsilon, E\upsilon, E\upsilon^*)) + r$$

$$\leq \left(\phi\left(4b^5 S_b(E\upsilon, E\upsilon, E\upsilon^*)\right) + r\right)^{\alpha(\upsilon, \upsilon, E\upsilon)\beta(\upsilon^*, \upsilon^*, E\upsilon^*)}$$

$$\leq \Omega\left(\phi\left(N_E{}^4(\nu, \upsilon^*)\right)\right)\phi\left(N_E{}^4(\nu, \upsilon^*)\right) + r$$

Where:

$$N_E{}^4(\upsilon, \upsilon^*) = \max\left\{\begin{array}{c} S_b(\upsilon, \upsilon, \upsilon^*), S_b(\upsilon, \upsilon, E\upsilon), S_b(\upsilon^*, \upsilon^*, E\upsilon^*) \\ \dfrac{S_b(\upsilon, \upsilon, E\upsilon^*)S_b(\upsilon^*, \upsilon^*, E\upsilon)}{1 + S_b(\upsilon, \upsilon, \upsilon^*) + S_b(E\upsilon, E\upsilon, E\upsilon^*)} \end{array}\right\} = S_b(\upsilon, \upsilon, \upsilon^*)$$

$$\phi(4b^5 S_b(\upsilon, \upsilon, \upsilon^*)) \leq \Omega(\phi(N_E{}^4(\upsilon, \upsilon^*))(\phi(S_b(\upsilon, \upsilon, \upsilon^*))) < \phi(S_b(\upsilon, \upsilon, \upsilon^*))$$

Which is contradiction. Unless $S_b(\nu, \nu, \nu^*) = 0$, that is $\nu = \nu^*$. Hence E has a unique fixed point.

Corollary: In the hypothesis of above Theorem, replace i = 3 in place of i = 4. Then, E has a unique fixed point.

Example: Let $S_b: X \times X \times X \to R^+$ be a mapping defined as:

$$S_b(p, q, r) = (|q+r-2p|+|q-r|)^2$$

where, X = [0, ∞) and ⪯ by p⪯r⇔ p≤r. So clearly (X, S_b, ⪯) is complete ordered S_b-metric space with b = 4. Define E: X→X by:

$$E(p) = \begin{cases} \frac{p^2}{4^8}, p \in [0,1] \\ 3p, p \in (1, \infty) \end{cases}$$

Also define α, β: X×X×X→R^+ and Ω: [0, ∞]→[0, 1), ϕ: [0, ∞)→[0, ∞) as:

$$\alpha(p, p, q) = \beta(p, p, q) = \begin{cases} 1, (p, p, q) \in [0, 1] \\ 0, \text{Otherwise} \end{cases}$$

and:

$$\Omega(t) = \frac{1}{4^2}, \phi(t) = t$$

Since (X, S_b, ⪯) is complete ordered S_b-metric space. We show that E is an (α, β)-admissible mapping. Let p, q∈ X, if α(p, p, q)≥1 and β(p, p, q)≥1 then p, q∈ [0,1]. On the other hand, for all p ∈ [0,1] then E(p)≤1. It follows α(Ep, Ep, Eq)≥1 and β(Ep, Ep, Eq)≥1. Therefore, the predication holds. In support of the above argument α(0, 0, E0)≥1 and β(0, 0, E0)≥1. Now, if {p_n} is a sequence in X such that α(p_n, p_n, p_{n+1})≥1 and β(p_n, p_n, p_{n+1})≥1 and p_n →p∈ X, for all n∈N∪ {0}, then p_n ⊆ [0, 1] and hence p∈ [0, 1]. This implies α(p, p, Ep)≥1 and β(p, p, Ep)≥1 . Let p, q∈ [0, 1]. Then:

$$\left(\phi\left(4b^5 S_b(Ep, Ep, Eq)\right) + r\right)^{\alpha(p, p, Ep)\beta(q, q, Eq)} = \phi\left(4b^5 S_b(Ep, Ep, Eq)\right) + r$$

$$= 4b^5(|Ep + Eq - 2Ep| + |Ep - Eq|)^2 + r$$

$$= 4b^5\left(2\left|\frac{p^2}{4^8} - \frac{q^2}{4^8}\right|\right)^2 + r$$

$$\leq \frac{1}{4^2}(2|p-q|)^2 \leq \frac{1}{4^2} S_b(p, p, q) + r$$

$$\leq \Omega(\phi(N_E{}^4(p,q))(\phi(N_E{}^4(p,q)) + r$$

Hence, the given inequality is satisfied. Otherwise α(p, p, Ep) β(q, q, Eq) = 0. Then:

$$(\phi(4b^5 S_b(Ep, Ep, Eq)) + r)^{\alpha(p,p,Ep)\beta(q,q,Eq)} =$$
$$1 \leq \Omega\left(\phi(S_b(p,\ p, q)\right)\left(\phi(S_b(p,\ p, q)\right) + r$$

Therefore, all the conditions are satisfied of above Theorem and 0 is unique fixed point of E.

Theorem: Let (X, S_b, ⪯) is complete ordered S_b-metric space, E: X→X be a mapping satisfies: (I) S_b (Ex, Ex, Ey)⪯Ω (S_b (x, x, y)) S_b (x, x, y) for all x, y ∈ X.

(II) E is continuous or if an increasing sequence {x_n}→x∈ X, then x_n⪯x ∀ n∈N. Further if x_0 ∈X with x_0⪯Ex_0 E. Then, E has a unique fixed point in X.

Proof: Similar proof follows from above Theorem.

Example: Let S_b: X×X×X → R^+ be a mapping defined as:

$$S_b(p, q, r) = (|q+r-2p|+|q-r|)^2$$

where, X= [0, ∞) and ⪯ by p⪯r⇔ p≤r. So clearly (X, S_b, ⪯) is complete ordered S_b-metric space with b = 4. Define E: X→X by:

$$E(p) = \frac{1}{4} p^2$$

for all p∈ X, also define Ω: [0, ∞)→[0, 1), by:

$$\Omega(t) = \frac{1}{2}$$

Then, by above Theorem, 0 is unique fixed point of E.

Theorem: In the hypotheses of above Theorem, replace (2) in place of (1). Then, E has a unique fixed point.

Corollary: In the hypotheses of above Theorem, replace i = 3 in place of i = 4. Then, E has a unique fixed point.

APPLICATIONS

Application to homotopy

Theorem: Let (X, S_b) be complete S_b-metric space, U and Ū be an open and closed subset of X such that U ⊆Ū. Assume that α, β: X×X×X→$\mathbb{R}^+$, H_b: Ū×[0,1] → X be an (α, β)-admissible operator satisfying the following conditions:

- u≠,Hb (u, κ) for each u∈ ∂U and κ ∈ [0,1] (Here ∂U is boundary of U in X)
- α (u, u, H_b (u, κ)) β(v, v, H_b (v, κ))ϕ(4b^5 S_b(H_b(u, κ), H_b(u, κ), H_b(u, κ))
- ≤Ω (ϕ (S_b(u, u, v)) ϕ(S_b(u ,u, v))

For all u, v$\in$ Ū and $\kappa \in [0, 1]$, where $\Omega\in\Omega$ and $\phi\in\Phi$:

- There exist $M_b \geq 0$ such that $S_b (H_b (u, \kappa), H_b (u,\kappa), H_b (u, \zeta)) \leq M_b|\kappa-\zeta|$. For every $u \in$ Ū and $\kappa, \zeta \in [0, 1]$. Then, H_b (.,0) has a fixed point $\Leftrightarrow$ H_b (.,1) has a fixed point

Proof: Let the set $B = \{\kappa \in [0, 1]: u = H_b (u, \kappa)$ for some $u \in U\}$. Since H_b (.,0) has a fixed point in U, so $0 \in B$.

Now, prove B is closed as well as open in [0, 1] and hence by the connectedness B = [0, 1]. Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq B$ with $\kappa_n \to \kappa \in [0, 1]$ as $n \to\infty$.

Now, $\kappa \in B$ must be shown. Since $\kappa_n \in B$ for n = 0, 1, 2, 3,$\cdots$.. there exists $u_n \in U$ with $u_{n+1} = H_b (u_n, \kappa_n)$. Since H_b is (α, β)-admissible operator:

$$\alpha(u_0, u_0, H_b (u_0, \kappa_0)) = \alpha(u_0, u_0, u_1) \geq 1$$

$$\alpha(H_b (u_0, \kappa_0), H_b (u_0, \kappa_0), H_b (u_1, \kappa_1) = \alpha (u_1, u_1, u_2) \geq 1$$

and:

$$\alpha(H_b (u_1, \kappa_1), H_b (u_1, \kappa_1), H_b (u_2, \kappa_2) = \alpha (u_2, u_2, u_3) \geq 1$$

Hence by induction $\alpha(u_n, u_n, u_{n+1}) \geq 1$ for all $n \geq 0$. Similarly, $\beta(u_n, u_n, u_{n+1}) \geq 1$ for all $n \geq 0$. Consider:

$$S_b (u_{n+1}, u_{n+1}, u_{n+2}) = S_b (H_b (u_n, \kappa_n), H_b (u_n, \kappa_n), H_b (u_{n+1}, \kappa_{n+1}))$$

$$\leq 2bS_b (H_b (u_n, \kappa_n), H_b (u_n, \kappa_n), H_b (u_{n+1}, \kappa_n))$$

$$+b^2S_b (H_b (u_{n+1}, \kappa_n), H_b (u_{n+1}, \kappa_n), H_b (u_{n+1}, \kappa_{n+1}))$$

$$\leq 2bS_b (H_b (u_n, \kappa_n), H_b (u_n, \kappa_n), H_b (u_{n+1}, \kappa_n))+b^2M |\kappa_n-\kappa_{n+1}|$$

Letting $n\to\infty$:

$$\lim_{n\to\infty} S_b (u_n, u_n, u_{n+1}) \leq \lim_{n\to\infty} 2bS_b (H_b (u_n, \kappa_n), H_b (u_n, \kappa_n), H_b (u_{n+1}, \kappa_n)) + 0$$

By the hypothesis, we have:

$$\lim_{n\to\infty} \phi(2b^4S_b (u_{n+1}, u_{n+1}, u_{n+2}))$$

$$\leq \lim_{n\to\infty} [\alpha(u_n, u_n, H_b (u_n, \kappa_n) \beta(u_{n+1}, u_{n+1}, H_b (u_{n+1}, \kappa_n))]$$

$$\phi(4b^5S_b (H_b (u_n, \kappa_n), H_b (u_n, \kappa_n), H_b (u_{n+1}, \kappa_n)))$$

$$\leq \lim_{n\to\infty} \Omega(\phi(S_b (u_n, u_n, u_{n+1}))), \phi(S_b (u_n, u_n, u_{n+1}))$$

Therefore:

$$\frac{\lim_{n\to\infty} \phi(2b^4S_b (u_{n+1}, u_{n+1}, u_{n+2}))}{\lim_{n\to\infty} \phi(S_b (u_n, u_n, u_{n+1}))} \leq \lim_{n\to\infty} \Omega(\phi(S_b (u_n, u_n, u_{n+1}))) < 1$$

In the above Inequality:

$$\leq \lim_{n\to\infty} \Omega(\phi(S_b (u_n, u_n, u_{n+1}))) = 1$$

Since $\Omega\in\Omega$, it follows:

$$\lim_{n\to\infty} \phi(S_b (u_n, u_n, u_{n+1})) = 0$$

Which yields:

$$\lim_{n\to\infty} (S_b (u_n, u_n, u_{n+1})) = 0 \tag{7}$$

Now, $\{u_n\}$ is a S_b-Cauchy sequence in (X, S_b) is to be shown. On contrary assume that $\{u_n\}$ is not a S_b-Cauchy sequence.

There is an $\varepsilon>0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k>m_k$:

$$S_b (u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon \tag{8}$$

and:

$$S_b (u_{m_k}, u_{m_k}, u_{n_{k-1}}) < \varepsilon \tag{9}$$

Therefore, from Eq. 8 and 9:

$$\varepsilon \leq S_b (u_{m_k}, u_{m_k}, u_{n_k})$$

$$\leq 2bS_b (u_{m_k}, u_{m_k}, u_{m_{k+1}}) + b^2S_b (u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k})$$

Letting $k\to\infty$:

$$4b^3 \in \leq \lim_{n\to\infty} 4b^5S_b (u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}) \tag{10}$$

But:

$$\lim_{n\to\infty} \phi(4b^5S_b (u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}))$$

$$\leq \lim_{n\to\infty} \left\{\alpha(u_{m_k}, u_{m_k}, H_b\left(u_{m_k}, \kappa_{m_k}\right)\right)\beta(u_{n_k\text{-}1}, u_{n_k\text{-}1}, H_b\left(u_{n_k\text{-}1}, \kappa_{n_k\text{-}1}\right)\left.\right)\right\}$$

$$\phi\left(4b^5S_b\left(H_b\left(u_{m_k},\kappa_{m_k}\right),H_b\left(u_{m_k},\kappa_{m_k}\right),H_b\left(u_{n_{k-1}},\kappa_{n_{k-1}}\right)\right)\right)$$

$$\leq \lim_{n\to\infty}\Omega\left(\phi\left(S_b(u_{m_k},u_{m_k},u_{n_{k-1}})\right)\right)\phi\left(S_b(u_{m_k},u_{m_k},u_{n_k})\right)$$

From Eq. 10:

$$\left(4b^3\in\right)\leq \lim_{n\to\infty}\phi\left(4b^5S_b\left(u_{m_{k+1}},u_{m_{k+1}},u_{n_k}\right)\right)$$

$$\leq \lim_{n\to\infty}\Omega\left(\phi\left(S_b(u_{m_k},u_{m_k},u_{n_{k-1}})\right)\right)\phi\left(S_b(u_{m_k},u_{m_k},u_{n_k})\right)$$

$$\leq \lim_{n\to\infty}\Omega\left(\phi\left(S_b\left(u_{m_k},u_{m_k},u_{n_k-1}\right)\right)\right)\left(4b^3\in\right)$$

That is:

$$1\leq \lim_{n\to\infty}\Omega\left(\phi\left(S_b\left(u_{m_k},u_{m_k},u_{n_k-1}\right)\right)\right)$$

Which implies:

$$\lim_{n\to\infty}\Omega\left(\phi\left(S_b\left(u_{m_k},u_{m_k},u_{n_k-1}\right)\right)\right)=1$$

Consequently, we obtain:

$$\lim_{n\to\infty}S_b\left(u_{m_k},u_{m_k},u_{n_k-1}\right)=0$$

and hence:

$$\lim_{n\to\infty}S_b\left(u_{m_k+1},u_{m_k+1},u_{n_k}\right)=0$$

It is a contradiction.

Hence $\{u_n\}$ is a S_b-Cauchy sequence in (X, S_b). By completeness there exists $\eta\in U$ such that:

$$\lim_{n\to\infty}u_n=\eta=\lim_{n\to\infty}u_{n+1} \qquad (11)$$

Now:

$$\phi\left(\frac{1}{2b}S_b\left(H_b(\eta,\kappa),H_b(\eta,\kappa),\eta\right)\right)\leq \lim_{n\to\infty}\inf\ \phi\left(S_b\left(H_b(\eta,\kappa),H_b(\eta,\kappa),H_b\left(u_n,\kappa\right)\right)\right)$$

$$\leq \liminf_{n\to\infty}\begin{pmatrix}\alpha(\eta,\eta.H_b(\eta,\kappa))\beta(u_n,u_n,H_b\left(u_n,\kappa\right))\\ \phi\left(4b^5S_b\left(H_b(\eta,\kappa),H_b(\eta,\kappa),H_b\left(u_n,\kappa\right)\right)\right)\end{pmatrix}$$

$$\leq \lim_{n\to\infty}\Omega\left(\phi(S_b(\eta,\eta,u_n))\right)\phi(S_b(\eta,\eta,u_n)$$

So:

$$\frac{\phi\left(\frac{1}{2b}S_b\left(H_b(\eta,\kappa),H_b(\eta,\kappa),\eta\right)\right)}{\lim_{n\to\infty}\phi(S_b(\eta,\eta,u_n)}\leq \lim_{n\to\infty}\Omega\left(\phi(S_b(\eta,\eta,u_n)\right)$$

That is:

$$1\leq \lim_{n\to\infty}\Omega\left(\phi(S_b(\eta,\eta,u_n)\right)$$

implies:

$$\lim_{n\to\infty}\Omega\left(\phi(S_b(\eta,\eta,u_n)\right)=1$$

Consequently, we get:

$$\lim_{n\to\infty}\phi(S_b(\eta,\eta,u_n)=0$$

and hence $S_b(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta) = 0$. It follows that $H_b(\eta, \kappa) = \eta$.

Thus $\kappa\in B$. Clearly B is closed in [0, 1]. Let $\kappa_0\in B$. Then there exists $u_0\in U$ such that $u_0 = H_b(u_0, \kappa_0)$. Since U is open, then there exists r>0 such that $B_{S_b}(u_0, r)\subseteq U$. Choose $\kappa\in(\kappa_0-\epsilon, \kappa_0+\epsilon)$ such that:

$$\left|\kappa-\kappa_0\right|\leq\frac{1}{M^n}<\epsilon$$

Then, for:

$$u\in\overline{B}_p(u_0, r) = \{u \in X/S_b(u, u, u_0)\underline{\leq} r+b^2S_b(u_0, u_0, u_0)\}$$

$$S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) = S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u_0, \kappa_0))$$

$$\underline{\leq}2bS_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \kappa_0))$$

$$+b^2S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0))$$

$$\underline{\leq}2bM|\kappa-\kappa_0|+b^2S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0))$$

Letting $n\to\infty$ and applying ϕ on both sides, then:

$$\varphi(S_b(H_b(u, \kappa), H_b(u, \kappa), u_0)\underline{\leq}\varphi(b^2S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)))$$

$$\leq \alpha(u, u, H_b(u, \kappa_0))\, \beta(u_0, u_0, H_b(u_0, \kappa_0))$$

$$\varphi(4b^5S_b(H_b(u, \kappa_0))\, H_b(u, \kappa_0, H_b(u_0, \kappa_0)))$$

$$\leq \Omega(\varphi(S_b(u, u, u_0)))\, \varphi(S_b(u, u, u_0)) \leq \varphi(S_b(u, u, u_0))$$

Therefore:

$$S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) < S_b(u, u, u_0) \leq r + b^2S_b(u_0, u_0, u_0)$$

Thus for each fixed $\kappa \in (\kappa_0-\epsilon, \kappa_0+\epsilon)$, $H_b(, ;\kappa): \bar{B}_p(u_0, r) \rightarrow \bar{B}_p(u_0, r)$. Then, all the conditions of Theorem (4.1) holds. Thus, we conclude that $H_b(. ;\kappa)$ has a fixed point in $\bar{U}$. But this must be in U. Therefore, $\kappa \in B$ for $\kappa \in (\kappa_0-\epsilon, \kappa_0+\epsilon)$. Hence$(\kappa_0-\epsilon, \kappa_0+\epsilon) \subseteq B$. Clearly B is open in [0, 1].

Similar process can be used to prove the converse.

Applications to integral equations

Theorem: Consider the I.V.P:

$$x'(t) = K(t, x(t));\ t \in I = [0, 1],\ x(0) = x_0 \tag{12}$$

where, $K: I \times R \rightarrow R$ is a continuous function and $x_0 \in R$. Let $\Omega: [0, \infty) \rightarrow [0, 1)$, $\phi: [0, \infty) \rightarrow [0, \infty)$ be a two functions defined as:

$$\Omega(t) = \frac{1}{3}, \phi(t) = t$$

And consider the following conditions:

- If there exist a function $\theta: R^3 \rightarrow R$ such that there is an $x_1 \in C(I)$, for all $t \in I$, we have:

$$\theta\left(x_1(t), x_1(t), \int_0^t K\left(s, x_1(s)\right)ds\right) \geq 0$$

- For all $t \in I$ and for all $x, y \in C(I)$, $\theta(x(t), x(t), y(t)) \geq 0 \Rightarrow$:

$$\theta\left(\frac{x_0}{3b^3} + \int_0^t K\left(s, x(s)\right)ds,\ \frac{x_0}{3b^3} + \int_0^t K\left(s, x(s)\right)ds,\ \frac{y_0}{3b^3} + \int_0^t K\left(s, y(s)\right)ds\right) \geq 0$$

- For any point x of a sequence $\{x_n\}$ of points in C(I) with:

$$\theta(x_n, x_n, x_{n+1}) \geq 0, \lim_{n\rightarrow\infty} \inf \theta(x_n, x_n, x) \geq 0$$

Then, (12) has a unique solution.

Proof: The integral equation of I.V.P (12) is:

$$x_0 + 3b^3\int_0^t K\left(s, x(s)\right)ds$$

Let X = C (I) be the space of all continuous functions defined on I and let $S_b(x, y, z) = (|y+z-2x|+|y-z|)^2$ for $x, y, z \in X$. Then (X, S_b) is a complete S_b-metric space, also define $E: X \rightarrow X$ by:

$$E(x)(t) = \frac{x_0}{3b^3} + \int_0^t K\left(s, x(s)\right)ds \tag{13}$$

Now:

$$(4b^5S_b(Ex(t), Ex(t), Ey(t))) = 4b^5\{|Ex(t)+Ey(t)-2Ex(t)|+|Ex(t)-Ey(t)|\}^2$$

$$= 16b^5|Ex(t)-Ey(t)|^2$$

$$= \frac{16b^5}{9b^6}\left|x_0 + 3b^3\int_0^t K\left(s, x(s)\right)ds - y_0 - 3b^3\int_0^t K\left(s, y(s)\right)ds\right|^2$$

$$\frac{16}{9b}\left|x(t) - y(t)\right|^2$$

$$\leq \frac{1}{3}S(x, x, y) \leq \Omega(\phi(N_E^4(x, y)))\, \phi(N_E^4(x, y))$$

Thus:

$$(4b^5S_b(Ex(t), Ex(t), Ey(t)) \leq \Omega((N_E^4(x, y)))\, (N_E^4(x, y))\ \forall x, y \in X$$

With $\theta(x(t), y(t)) \geq 0$ for all $t \in I$. Define $\alpha, \beta: X \times X \times X \rightarrow [0, \infty)$ by:

$$\alpha(x, x, y) = \beta(x, x, y) = \begin{cases} 1, \theta(x(t), x(t), y(t)) \geq 0, t \in I \\ 0, \text{Otherwise} \end{cases}$$

Then, obviously E is an (α, β)-admissible, for all, $y \in X$, then:

$$\alpha(x, x, Ex)\ \beta(y, y, Ey)\ (4b^5S_b(Ex(t), Ex(t), Ey(t))) \leq \Omega((N_E^4(x, y)))\, (N_E^4(x, y))$$

It follows from Eq. 2, E has a unique fixed point in X.

CONCLUSION

This study presents some fixed point results by using (α, β)-admissible Geraghty type rational contractive conditions defined on ordered S_b-metric spaces and suitable examples that supports the main results. Also, applications to Homotopy theory as well as integral equations are provided.

SIGNIFICANCE STATEMENT

This study proposed a framework to established fixed point results by using (α, β)-admissible Geraghty type rational contractions in ordered S_b-metric spaces. This study will help researchers to generalized different contractions in S_b-metric spaces with applications to integral equations as well as Homotopy theory. Thus, a new framework on S_b-metric spaces may be arrived at.

ACKNOWLEDGMENT

The authors are very thankful to the reviewers and editors for their valuable comments, remarks and suggestions which improved the paper in good form.

REFERENCES

1. Banach, S., 1922. Sur les operations dans les ensembles abstraits et leurs application aux equations integrales. Fund. Math., 3: 133-181.
2. Banach, S., 1932. Theorie des Operations Lineaires. Manograic Mathematic Zne, Warsaw, Poland, Pages: 254.
3. Ansari, A.H., O. Ege and S. Randenovic, 2017. Some fixed point results on complex valued G_b-metric spaces. Rev. Real Acad. Cienc. Exactas Fisicas Natl. Serie A. Matemat., 112: 463-472.
4. Chandok, S., B.S. Choudhury and N. Metiya, 2015. Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions. J. Egypt. Math. Soc., 23: 95-101.
5. Mustafa, Z., M.M.M. Jaradata and H.M. Jaradat, 2016. Some common fixed point results of graphs on b-metric space. J. Nonlinear Sci. Applic., 9: 4838-4851.
6. Sedghi, S., A. Gholidahneh, T. Dosenovic, J. Esfahani and S. Radenovic, 2016. Common fixed point of four maps in S_b-metric spaces. J. Linear Topol. Algebra, 5: 93-104.
7. Nizar, S. and N. Mlaiki, 2016. A fixed point theorem in S_b-metric spaces. J. Math. Comput. Sci., 16: 131-139.
8. Souayah, N., 2016. A fixed point in partial S_b-metric spaces. Analele Univ. Ovidius Constanta-Seria Matemat., 24: 315-362.
9. Rohen, Y., T. Dosenovic and S. Randanovic, 2017. A note on the paper "A fixed point theorems in S_b-metric spaces". Filomat, 31: 3335-3346.
10. Kishore, G.N.V., K.P.R. Rao, D. Panthi, B.S. Rao and S. Satyanaraya, 2017. Some applications via fixed point results in partially ordered S_b-metric spaces. Fixed Point Theory Applic., Vol. 2017. 10.1186/s13663-017-0603-2.
11. Geraghty, M.A., 1973. On contractive mappings. Proc. Am. Math. Soc., 40: 604-608.
12. Samet, B., C. Vetro and P. Vetro, 2012. Fixed point theorems for α—ψ-contractive type mappings. Nonlinear Anal.: Theory Methods Applic., 75: 2154-2165.
13. Cho, S.H., J.S. Bae and E. Karapinar, 2013. Fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Applic., Vol. 2013. 10.1186/1687-1812-2013-329.
14. Karapinar, E., 2014. - -Geraghty contraction type mappings and some related fixed point results. Filomat, 28: 37-48.
15. Chandok, S., 2015. Some fixed point theorems for (α, β)-admissible Geraghty type contractive mappings and related results. Math. Sci., 9: 127-135.
16. Gupta, V., W. Shatanawi and N. Mani, 2016. Fixed point theorems for (ψ,β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations. J. Fixed Point Theory Applic., 19: 1251-1267.
17. Hussain, N., E. Karapınar, P. Salimi and F. Akbar, 2013. α-admissible mappings and related fixed point theorems. J. Inequalities Applic., Vol. 2013. 10.1186/1029-242X-2013-114.
18. Abbas, M. and M. Doric, 2010. Common fixed point for generalized (α,β)-weak contractions. Math. Un Nis. Serbia, 10: 1-10.
19. Mustafa, Z., M.M. Jaradat, A.H. Ansari, B.Z. Popovic and H.M. Jaradat, 2016. C-class functions with new approach on coincidence point results for generalized (ψ,ϕ)-weakly contractions in ordered b-metric spaces. Springer Plus, 10.1186/s40064-016-2481-1.
20. Murthy, P.P., K. Tas and U.D. Patel, 2015. Common fixed point theorems for generalized (φ,ψ) -weak contraction condition in complete metric spaces. J. Inequalities Applic., Vol. 2015. 10.1186/s13660-015-0647-y.
21. Nashine, H.K. and B. Samet, 2011. Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal.: Theory Methods Applic., 74: 2201-2209.
22. Roshan, J.R., V. Parvaneh, S. Radenovic and M. Rajovic, 2015. Some coincidence point results for generalized (ψ,ϕ)-weakly contractions in ordered *b*-metric spaces. Fixed Point Theory Applic., Vol. 2015. 10.1186/s13663-015-0313-6.
23. Arshad, M., Z. Kadelburg, S. Radenovic, A. Shoaib and S. Shukla, 2017. Fixed points of α-Dominated mappings on dislocated quasi metric spaces. Filomat, 31: 3041-3056.
24. Zhou, M., X.L. Liu and S. Radenovic, 2017. S-γ-ϕ-φ-contractive type mappings in S-metric spaces. J. Nonlinear Sci. Applic., 10: 1613-1639.
25. Jaradat, M.M.M., Z. Mustafa, A.H. Ansari, P.S. Kumari, D. Dolicanin-Djekic and H.M. Jaradat, 2017. Some fixed point results for $F_{\alpha-\omega\varphi}$-generalized cyclic contractions on metric-like space with applications to graphs and integral equations. J. Math. Anal., 8: 28-45.