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Static Analysis of Non-Uniform Elastic Composite Beams

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Abstract: In this paper, a static analysis using Transfer Matrix – Analog Beam Method is presented for elastic composite beams. The expression of the field transfer matrix for a beam element subjected to any type of static loading is derived. The effects of static point load, intermediate supports and shear transmitting studs are considered. Results of the present model are compared with the existing results to verify the validity of the model. The application of the model is demonstrated by investigating the static characteristics of composite steel concrete beams, such as bridges. Finally, using the presented model, the results are presented shown the variation of the geometric and material parameters of a non-uniform beam subjected to a distributed load.

Key words: Numerical models, Transfer matrix-analog beam method, Static analysis, Non-uniform elastic composite beams

Introduction

A method of analysis for beams with shear-lag effects from wide flanges or with shear deformation resulting from shear studs in composite beams was developed by Gjelsvik (1991). The basic idea of the method was to replace the real beam with an analog beam, where all shear deformation is concentrated in a thin layer called shear layer. An application of the model of the analog beam for the static analysis of a simply supported beam with uniformly distributed load was presented by Betti *et al.* (1996).

A mathematical model of elastic composite beam based on the combination of transfer matrix method and analog beam method was developed by Ellakany *et al.* (2000; 2001 and 2001). An exact field transfer matrix for elastic composite beam element with uniformly distributed mass was obtained Ellakany (2000) and used to study the free vibration of simply supported beams. The model was developed to predict the natural frequencies of multi-span elastic composite beams, the effect of rigid support have been included in addition to the effect of shear restraint at both supports Ellakany *et al.* (2001). Also, an extension was made to take into account the effect of intermediate conditions, such as elastic support and hinge, on the free vibration of elastic composite beam Ellakany *et al.* (2001).

The static and dynamic behavior of stiffened plates was presented by Bedair (1997). The stiffened plates are encumbered in a bridge deck, which is composed of slab or plate element reinforced by beam element located at discrete distance in both longitudinal and transversal directions. A closed - form solution for reinforced Timoshinko beam on an elastic beam subjected to any pressure loading was obtained by Yin (2000).

The main objective of this study is to present a numerical model based on transfer matrix- analog beam method for non-uniform elastic composite beams subjected to any type of static loading. The model takes into account both-point and distributed loads and one and multi-span beams. So, the model covers most types of elastic composite beams and static loadings. A particular case of two spans non-uniform elastic composite beam subjected to uniform distributed load is considered to study the effect of beam parameters, such as stiffness of shear layer and bending stiffness ratio, on the static behavior of that beam. The model could be improved to derive the exact dynamic field matrix, which will be used to study the dynamic response of elastic composite beam subjected to various types of dynamic loading.

Numerical Model

Extended field matrix: One of the most common types of elastic composite beam is the composite concrete beams that are composed of concrete slab and steel beam. The slab and beam are connected at their interface by a shear-transmitting device such as studs. Fig.1 shows a typical type of elastic composite beam and its coordinate system. Consider a beam element of length (l) subjected to a static load (P) per unit length. The displacements and forces scheme for the beam element are shown in Fig.2. For easy reference, the basic equations of bending moments and shear force for an elastic composite beam are presented in Ellakany *et al.* (2000). The equation of total bending moment for the elastic composite beam element shown in Fig. 1 is:

$$M = \frac{EI_t}{kh^2}\frac{dQ}{dx} + \frac{EI_tEI_c}{kh^2}\frac{d^4w}{dx^4} - EI\frac{d^2w}{dx^2} \qquad (1)$$

Where

$$EI = EI_t + EI_c$$

For the case of a beam element subjected to a distributed load P, the equilibrium considerations give the equations

$$\frac{dQ}{dx} = -P \qquad ,Q = \frac{dM}{dx} \qquad (2)$$

Taking the second derivative of equation (1) with respect to x and combining with equation (2), the governing equation is expressed as

$$\frac{d^6w}{dx^6} - \mu\frac{d^4w}{dx^4} = \nu(P - \frac{EI_t}{kh^2}\frac{d^2P}{dx^2}) \qquad (3)$$

Where

$$\mu^2 = \frac{kh^2 EI}{EI_c\ EI_t} \quad \text{and}$$

$$v = -\frac{kh^2}{EI_c\ EI_t}$$

The distributed load may be, in general, a function of x, that is

$$P = f(x) \quad \text{for} \quad 0 < x < \ell \tag{4}$$

Where f(x) can be expressed in a Fourier cosine series as

$$P = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{\ell}x \tag{5}$$

Where

$$A_0 = \frac{1}{\ell}\int_0^{\ell} f(x)dx \quad \text{and}$$

$$A_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi}{\ell}x\,dx$$

Substituting into the governing equation (3) gives

$$\frac{d^6 w}{dx^6} - \mu^2 \frac{d^4 w}{dx^4} = v(A_0 + \sum_{n=1}^{\infty} \overline{A}_n \cos\frac{n\pi}{\ell}x) \tag{6}$$

Where

$$\overline{A}_n = 1 + \frac{EI_t}{kh^2}(\frac{n\pi}{\ell})^2$$

The solution to equation (6) is

$$w(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 \cosh\mu x + C_6 \sinh\mu x + B_0 \frac{x^4}{24} + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi}{\ell}x \tag{7}$$

Where

$$B_0 = \frac{v}{-\mu^2} A_0'$$

$$B_n = \frac{-A_n}{\mu^2(\frac{n\pi}{\ell})^4 + (\frac{n\pi}{\ell})^6}$$

and C_1, C_2, ... and C_6 are six constants.

The equations for angle of shear rotation, and moments are:

$$\varphi(x) = \frac{1}{kh^2}Q(x) - \frac{1}{kh^2}\frac{dM_c}{dx} - \frac{dw}{dx} \tag{8-a}$$

$$M_t(x) = EI_t \cdot \frac{d\varphi}{dx} \tag{8-b}$$

and

$$M_c(x) = -EI_c \cdot \frac{d^2 w}{dx^2} \tag{8-c}$$

Substitution of equation (7) into the above relations gives the following expressions for the displacements and forces

$$w'(x) = C_2 + C_3(2x) + C_4(3x^2) + C_5(\mu\sinh\mu x) + C_6(\mu\cosh\mu x) + B_0\frac{x^3}{6} - \sum_{n=1}^{\infty}\frac{n\pi}{\ell}B_n \sin\frac{n\pi}{\ell}x, \tag{9-a}$$

$$\varphi(x) = -C_2 - C_3(2x) - C_4(3x^2 + \frac{6EI_t}{kh^2}) + C_5(\alpha_1\sinh\mu x) + C_6(\alpha_1\cosh\mu x) - \frac{EI_t}{kh^2}B_0 x - B_0\frac{x^3}{6} + \sum_{n=1}^{\infty}\alpha_2 B_n \sin\frac{n\pi}{\ell}x \tag{9-b}$$

$$M_t(x) = EI_t[-2C_3 + C_4(6x) + C_5(\mu\alpha_1\cosh\mu x) + C_6(\mu\alpha_1\sinh\mu x) - \frac{EI_t}{kh^2}B_0 - B_0\frac{x^2}{2} + \sum_{n=1}^{\infty}\frac{n\pi}{\ell}\alpha_2 B_n \cos\frac{n\pi}{\ell}x] \tag{9-c}$$

$$M_c(x) = -EI_c[2C_3 + C_4(6x) + C_5(\mu^2\cosh\mu x) + C_6(\mu^2\sinh\mu x) + B_0\frac{x^2}{2} - \sum_{n=1}^{\infty}\left(\frac{n\pi}{\ell}\right)^2 B_n \cos\frac{n\pi}{\ell}x] \tag{9-d}$$

$$Q(x) = C_4(-6EI) + C_5(\alpha_3\sinh\mu x) + C_6(\alpha_3\cosh\mu x) - EI\ B_0 x + \sum_{n=1}^{\infty}\alpha_4 B_n \sin\frac{n\pi}{\ell}x \tag{9-e}$$

Where

$$\alpha_1 = \frac{\alpha_3}{kh^2} + \frac{EI_c}{kh^2}\mu^3 - \mu, \quad \alpha_2 = \frac{\alpha_4}{kh^2} + \frac{EI_c}{kh^2}(\frac{n\pi}{\ell})^3 + \frac{n\pi}{\ell}$$

$$\alpha_3 = \frac{EI_t EI_c}{kh^2}\mu^5 - EI\mu^3$$

$$\alpha_4 = \frac{n\pi}{\ell}\cdot\frac{EI_t EI_c}{kh^2}\left[\frac{A_n}{EI_c} - \frac{EI\ Kh^2}{EI_t EI_c}(\frac{n\pi}{\ell})^2 - (\frac{n\pi}{\ell})^4 B_n\right]$$

The above relations and equation (7) can be rewritten in a matrix form as

$$\begin{bmatrix} w \\ w' \\ f \\ M_t \\ M_c \\ Q \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 & \cosh\mu x & \sinh\mu x & a_7 \\ 0 & 1 & 2x & 3x^2 & \mu\sinh\mu x & \mu\cosh\mu x & a_8 \\ 0 & -1 & -2x & -\frac{6EI_1}{kh^2} - 3x^2 & a_1\sinh\mu x & a_1\cosh\mu x & a_9 \\ 0 & 0 & -2EI_t & -6EI_t x & a_5\cosh\mu x & a_5\sinh\mu x & a_{10} \\ 0 & 0 & -2EI_c & -6EI_c x & a_6\cosh\mu x & a_6\sinh\mu x & a_{11} \\ 0 & 0 & 0 & -6EI & a_3\sinh\mu x & a_3\cosh\mu x & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ 1 \end{bmatrix}$$

or

$$\mathbf{Z}(x)=\mathbf{D}(x).\mathbf{C} \qquad (10)$$

Where

$\alpha_5 = EI_t\mu\alpha_1$,

$\alpha_6 = -EI_c\mu^2$,

$$\alpha_7 = \frac{B_0x^4}{24} + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi}{\ell}x,$$

$$\alpha_8 = \frac{B_0x^3}{6} - \sum_{n=1}^{\infty} \frac{n\pi}{\ell} B_n \sin\frac{n\pi}{\ell}x,$$

$$\alpha_9 = -(\frac{EI_t}{kh^2}x + \frac{x^3}{6})B_0 + \sum_{n=1}^{\infty} \alpha_2 B_n \sin\frac{n\pi}{\ell}x,$$

$$\alpha_{10} = -EI_t[(\frac{EI_t}{kh^2} + \frac{x^2}{2})B_0 - \sum_{n=1}^{\infty} \frac{n\pi}{\ell}\alpha_2 B_n \cos\frac{n\pi}{\ell}x],$$

$$\alpha_{11} = -EI_c[\frac{B_0x^2}{2} - \sum_{n=1}^{\infty} (\frac{n\pi}{\ell})^2 B_n \cos\frac{n\pi}{\ell}x],$$

and

$$\alpha_{12} = -EI \cdot B_0x + \sum_{n=1}^{\infty} \alpha_4 B_n \sin\frac{n\pi}{\ell}x.$$

For the beam element **j** between two nodes **i** and **i+1**, Fig.3, the state vectors at both sides are $\mathbf{Z}_{i-1}^{R}$ for x=0 and $\mathbf{Z}_{i}^{L}$ for x=l, then equation (10) can be written as

$$\mathbf{Z}_{i-1}^{R} = \mathbf{Z}(0) = \mathbf{D}(0)\cdot\mathbf{C}$$

$$\mathbf{Z}_{i}^{L} = \mathbf{Z}(\ell) = \mathbf{D}(\ell)\cdot\mathbf{C}$$

Eliminating the vector of constants C from the above two equations gives

$$\mathbf{Z}_{i}^{L} = \mathbf{FM}_j.\mathbf{Z}_{i-1}^{R} \qquad (11)$$

The matrix FM_j between the state vectors $\mathbf{Z}_{i}^{L}$ and $\mathbf{Z}_{i-1}^{R}$ is called extended field matrix of beam element **j.**

Point matrix: The concentrated static load F is acting on node i, the displacements w, w' and φ and moments M_t and M_c are equal at both sides of node i, while the shear force Q is changed with the value of concentrated load Fi. The relations of the state vector $\mathbf{Z}_{i}^{R}$, at the right side of node i, and $\mathbf{Z}_{i}^{L}$, at the left side of node i, can be written in the matrix form as:

$$\begin{bmatrix} w \\ w' \\ \varphi \\ M_t \\ M_c \\ Q \\ 1 \end{bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -F \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_i^L$$

Or

$$\mathbf{Z}_{i}^{R} = \mathbf{PM}_i.\mathbf{Z}_{i}^{L} \qquad (12)$$

Where $\mathbf{PM}_i$ is known as extended point matrix.

Transfer matrix scheme: The actual non-uniform beam is modeled as a stepped beam by dividing it into N beam elements with uniform values of loads and stiffness, Fig.4. The properties of each element, such as load density, moment of inertia, shear stiffness and modulus of elasticity are calculated as mean values. Applying the transfer matrix method on the beam system shown in Fig.4, the relation between the state vectors at both ends S_N and S_0 of beam is:

$$Z_{S_N} = \mathbf{T}.Z_{S_0} \qquad (13)$$

Where **T** is called the overall transfer matrix which can be easily calculated by multiplying the field and point matrices, for example, the overall matrix for a simply supported beam with N elements and N+1 stations can be written as:

$$\mathbf{T}=\mathbf{FM}_N . \mathbf{PM}_N . \mathbf{FM}_{N-1} . \mathbf{PM}_{N-1} \ldots\ldots\ldots\ldots\ldots \mathbf{PM}_3 . \mathbf{FM}_2 .\mathbf{PM}_2 . \mathbf{FM}_1 \qquad (14)$$

The effect of intermediate conditions such as rigid and elastic supports can be taken into account when the overall transfer matrix has been calculated. For the case of intermediate rigid support at S_1, there is an internal unknown discontinuity that is the reaction R_1 at rigid support S_1. Corresponding to this unknown, we have the condition that the deflection w_{S0}=0.0. Applying this condition at support S_1, one of the initial unknowns of the state vector at S_1 is eliminated and the new unknown R_1 is introduced.

Boundary conditions at the end supports: After the transfer matrix scheme is completed and the overall transfer matrix has been computed the boundary conditions at the both ends of the beam system must be applied to obtain the unknown state vector elements at both ends. Then the displacements and forces at each node of beam system are calculated using matrix multiplication scheme. Two types of end boundary conditions can be considered, shear restraint at both ends and no shear restraint, these boundary conditions with various types of end support are summarized in Table 1.

Table 1: Boundary Conditions at End Supports

Type of end supports	Boundary conditions	
	With end shear restraint	No end shear restraint
Simple	$w=0.0$, $w'+\varphi=0.0$ and $M_t=-M_c$	$w=0.0$, $M_t=0.0$ and $M_c=0.0$
Fixed	$w=0.0$, $w'=0.0$ and $\varphi=0.0$	$w=0.0$, $\varphi=0.0$ and $M_c=0.0$
Free	$M_t=0.0$, $M_c=0.0$ and $EI\varphi''+(EI_tEI_c/kh^2)\varphi''''=0.0$	$M_t=0.0$, $M_c=0.0$ and $Q=0.0$

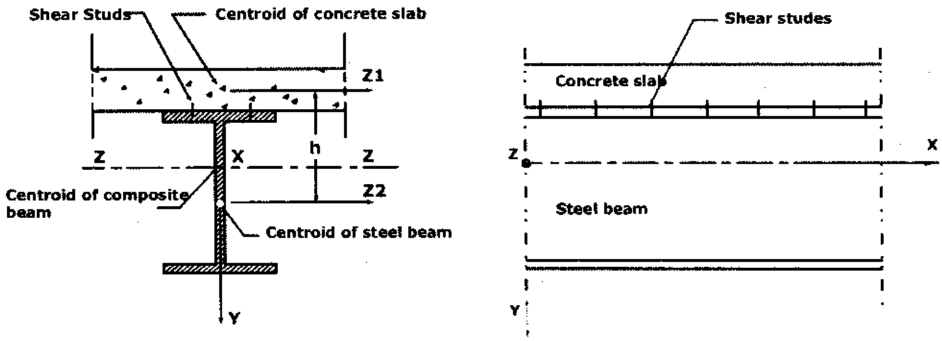

Fig. 1: Coordinate System of Elastic Composite Beam

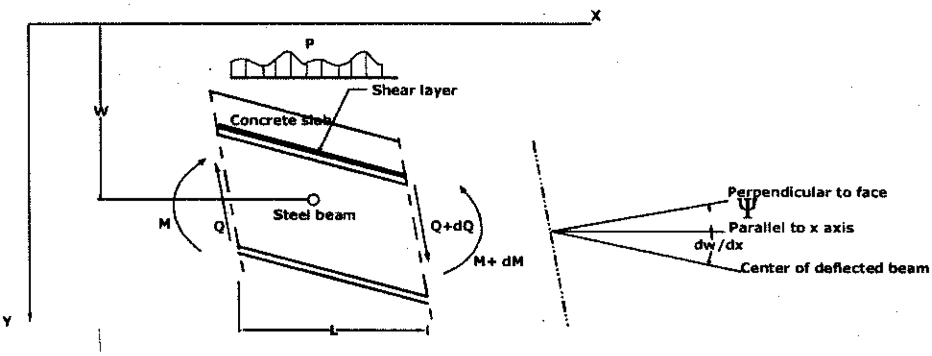

Fig. 2: Displacements and Forces of Elastic Composite Elements

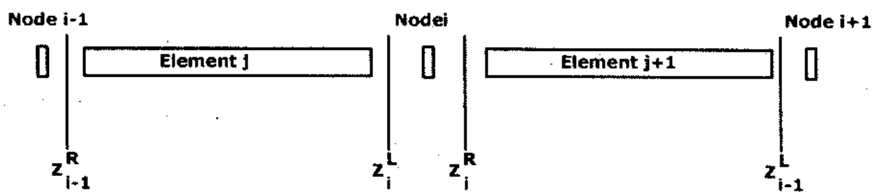

Fig. 3: State Vectors of Beam Elements and Nodes

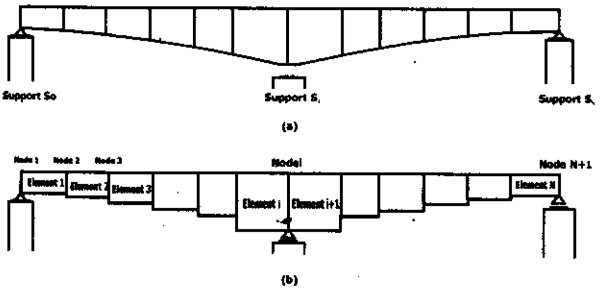

Fig.4 a) Actual beam system, b) Stepped beam system

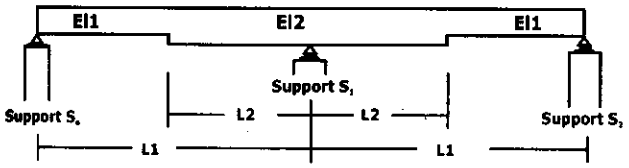

Fig.5: Two spans non-uniform beam with intermediate rigid support

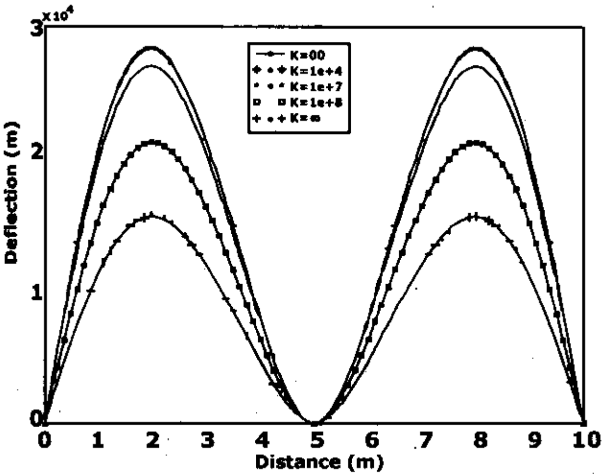

Fig.6: Static deflection of two spans elastic composite beam : no shear restraint Stiffness ratio=1.0

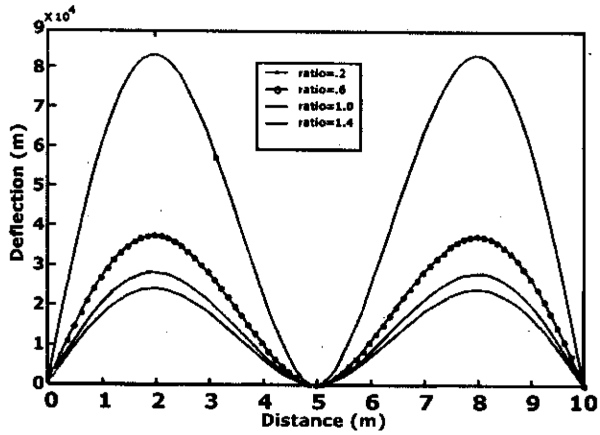

Fig.7: Static deflection of two spans elastic composite beam : no shear restraint Shear Stiffness ratio=1e+5

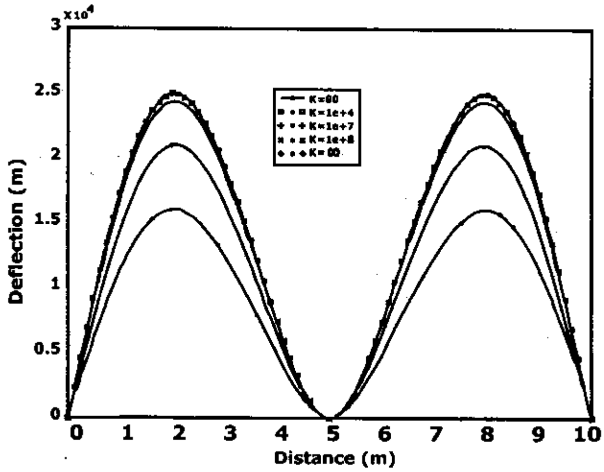

Fig.8: Static deflection of two spans elastic composite beam:with shear restraint Stiffness ratio=1e+5

Numerical Results

Verification of the method: To examine the accuracy of static solution obtained by the present method, the static deflection of a simply supported elastic composite bea subjected to uniform distributed load are calculated and compared with those obtained by Betti and Gjelsvk (1996). The data used for the beam and load are as follows: L=5ms, $EI=2x10^7 N.m^2$, h=0.7 m, An=0.0 (n=1,2,3....) And A_0=1000 N/m. Different values of shear stiffness k have been considered while an intermediate value of $EI_1= EI_c$ has been chosen. A good agreement is observed with all reported results in Table 2.

Parametric analysis of the Results :The presented method can be now used to calculate the static displacements and forces of elastic composite beams taken into account the various condition and parameters such as non-uniform beam, continuous beam, different intermediate conditions, and different types of loads. The following example is chosen to investigate the effects of the beam parameters on the static behavior of non-uniform elastic composite beam with 2 spans, Fig.5. The beam is simply supported at both ends. The data used in the analysis are as follows: L_1=5 m, L_2=2.5 m, $EI_1=2x10^7$ N/m^2, $EI_2=3x10^7$ N/m^2 and A_0=2000 N/m.

Table 2: Mid span deflections of simply supported beam

$\frac{EI_t}{kh^2}$	Deflection of mid span (m) Present method	R. Betti and A. Gjelsvik (1996)
0.00	0.413x10-3	0.407x10-3
0.10	0.538x10-3	0.540x10-3
0.25	0.628x10-3	0.629x10-3
0.50	0.693x10-3	0.695x10-3
1.00	0.743x10-3	0.745x10-3
∞	0.814x10-3	0.814x10-3

Different values of shear stiffness k have been considered to present the various degree of interaction between the upper slab and lower beam. Also, different values of bending stiffness ratio (λ=EIt/EIc) have been chosen to study the effect of bending stiffness for beam and truss components. Fig.6 presents the values of vertical deflections w(x) along the axis of the beam for various values of k. It is clear that the deflection changes from $0.29x10^{-3}$ for k=0.0 to $0.16x10^{-3}$ for k=∞, this change depends on the relation between the truss stiffness EI_t and bending stiffness EI_c. For the case of k=0.0 the effect of the truss stiffness is ignored and the beam is deflected as two sub-beam and so maximum deflection occurs. Otherwise, when the shear stiffness takes its normal values, the deflection is reduced until the minimum value for k=∞, and the beam behaves as the classical Navier beam. The effect of stiffness ratio λ on the deflection of elastic composite beam is shown in Fig.7. For a typical composite beam λ varies from about 0.2 to 1.4, the lower values corresponding to a thick slab and small beam, and the higher values to a thin slab and large beam. Fig.8 shows the deflections for the case of shear resistant at both ends with various values of shear stiffness k. Comparing between the deflections for the case of no shear restraint Fig.6 and with shear restraint Fig.8. It is found that for the case of k=∞ the deflections are the same, while for k=0.0 the maximum deflection reduces from $0.29x10^{-7}$m for the case of no shear restraint to $0.25x10^{-3}$ for the case with shear restraint. This is because when the shear restraint is removed the effect of bending truss is reduced and a large deflection occurs. The importance of carefully considering the sliding between the upper slab and the lower beam at the ends is evident.

Conclusion: In this paper, the transfer matrix-analog beam method is used to study the static behavior of non-uniform elastic composite beams. Both point load and any type of distributed load are included in the model which is one of the advantages compared with that presented by Betti (1996). The effects of the shear stiffness k and the bending stiffness ratio λ on the static behavior of non-uniform elastic composite beam are studied. The results show that static deflection of the elastic composite beam is strongly affected by both of the shear stiffness and the support conditions at the ends of the beam.

Nomenclature

l:	Length of beam element
M:	Total bending moment
M_t:	bending moment due to truss action
M_c:	bending moment.
EI:	total flexural rigidity of beam
EI_c:	Bending stiffness of beam component
EI_t:	Bending stiffness of truss component.
k:	Shear stiffness of shear layer
φ:	Rotation angle due to shear
w:	vertical displacement
h:	Distance between the centroid of upper slab and lower beam

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