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Modified Darcy's law to predict Low Reynolds Flow Through Porous Media

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Abstract: The flow rate of liquids through the porous media, as in slow deep filters or the column of packed resin beads, is of practically interest It has been well established that for low Reynolds number in flow through porous media, the pressure drop follows the Darcy's law. As low Reynolds number in flow through porous media resembles capillary flow due to surface tension forces, a key parameter looks to be neglected in Darcy's law. In the following analysis, using a capillary model to characterize the porous medium and assuming that each pore is a vertical cylindrical tube, a general form for low Reynolds flow through porous media has been developed. It can be shown that Darcy's law is valid only for very deep porous media.

Keywords: Darcy's law, Porous media

Introduction

The flow situation in the porous media is a complex one. One approach long used to model flow through porous media, has been to consider the medium as made up of bundles of straight capillaries or assemblages of randomly oriented straight pores of capillaries in which the flow is of poiseuille type.

It has been well established by experiments that for low Reynolds number flow through porous media, the pressure drop (dP/dx) follows Darcy's law [2].

$$\frac{dP}{dx} = -\frac{\mu}{k}u$$

where

μ = Viscosity of Liquid
u = Superficial Velocity

the constant k is called the permeability and has the dimensions of length squared (L^2).

The four principal approaches to determine relationship between the permeability k and the properties of the medium are: capillary model, drag model, orifice model and stochastic model [1,3-5]. All of these models shows that dimensionally $k{\sim}a^2$, where a is the characteristic radius of the pores. Therefore, Darcy's law predicts that the flow rate of liquids through porous media is proportional to $\frac{\rho g a^2}{\mu}$.

A key parameter in flow through porous media, like other surface phenomena, is surface tension. It seems that Darcy's law does not incorporate effects of surface tension on flow through porous media.

Theoretical Treatment: Using a capillary model to characterize the porous medium, the following analysis shows that a general form can be developed for low Reynolds flow through porous media. Assume each pore is a vertical cylindrical tube with the top of which is brought at time t = 0 into contact with a reservoir of incompressible liquid. Let a be the radius of the cylinder, z the distance of rise in capillary at time t, ρ the liquid density, μ its viscosity, and γ the surface tension of liquid.

We introduce cylindrical polar co-ordinates (r, ϕ, y) and corresponding fluid velocity components are v, o, and u respectively. Y is distance along the vertical axis into the tube and the origin o is on the axis of the tube at entry position (Fig. 1).

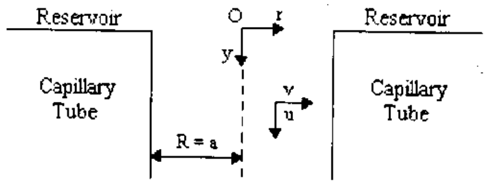

Fig. 1: The co-ordinate system for entry flow into a capillary tube

For an incompressible fluid the continuity equation is:

$$\frac{\partial}{\partial y}(ru) + \frac{\partial}{\partial r}(rv) = 0 \quad (1)$$

because by symmetry the flow conditions are independent of angle ϕ.

The Navier - Stokes equation for component u is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial y^2}\right]+g \quad (2)$$

where ν is the kinematics viscosity.

Multiplying equation (2) by r, yields:

$$r\frac{\partial u}{\partial t}+ru\frac{\partial u}{\partial y}+rv\frac{\partial u}{\partial r}=-\frac{r}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+r\frac{\partial^2 u}{\partial y^2}\right]+gr \quad (3)$$

By using continuity equation, equation (3) can be modified to equation (4)

$$\frac{\partial}{\partial t}(ru)+\frac{1}{2}\frac{\partial}{\partial y}(ru^2)+\frac{\partial}{\partial r}(ruv)=-\frac{r}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+r\frac{\partial^2 u}{\partial y^2}\right]+gr \tag{4}$$

Integrating with respect to r from r = 0 to r = a and using the no-slip condition we have

$$\frac{\partial}{\partial t}\int_0^a rudr+\frac{1}{2}\frac{\partial}{\partial y}\int_0^a ru^2dr=-\frac{1}{\rho}\int_0^a r\frac{\partial p}{\partial y}dr+\nu a\frac{\partial u}{\partial r}\Big|_{r=a}+\nu\frac{\partial^2}{\partial y^2}\int_0^a rudr+\frac{1}{2}ga^2 \tag{5}$$

It is obvious that:

$$\pi a^2\frac{dz}{dt}=2\pi\int_0^a rudr \tag{6}$$

and

$$\frac{\partial^2}{\partial y^2}\int_0^a rudr=\frac{\partial^2}{\partial y^2}\left(\frac{a^2}{2}\frac{dz}{dt}\right)=0$$

because z is a function of t only.
Therefore, equation (5) is reduced to equation (7):

$$2\frac{d^2z}{dt^2}+\frac{1}{2}\frac{\partial}{\partial y}\int_0^a ru^2dr=-\frac{1}{\rho}\int_0^a r\frac{\partial p}{\partial y}dr+\nu a\frac{\partial u}{\partial r}\Big|_{r=a}+\frac{1}{2}g \tag{7}$$

Assuming a poiseuille flow into capillary, we have

$$u=2\left(1-\frac{r^2}{a^2}\right)\frac{dz}{dt} \tag{8}$$

also:

$$\frac{\partial u}{\partial r}\Big|_{r=a}=-\frac{4}{a}\frac{dz}{dt} \tag{9}$$

As in poiseuille flow the radial velocity component, v, is equal to zero, therefore from v-component of the Navier – Stokes equation, one can find that p/r=0, hence p is independent of r, i.e., p=p(z,t).
Integrating equation (7) with respect to y from y=0 to y=z(t) gives

$$\frac{1}{2}a^2z\frac{d^2z}{dt^2}+\frac{1}{2}\left[\int_0^a ru^2dr\right]_{y=0}^{y=z(t)}=\frac{1}{\rho}\int_0^a[p(z,t)-p(o,t)]dr+\nu a\int_0^{z(t)}\left(\frac{\partial u}{\partial r}\right)_{r=a}dy+\frac{1}{2}ga^2z(t) \tag{10}$$

As u is independent of y, therefore,

$$\int_0^a ru^2dr\Big|_{y=0}^{y=z(t)}=0 \tag{11}$$

Using Laplace's equation yields

$$\mathbf{P(z,t)=P_0+}\frac{2\gamma\cos\theta}{a} \tag{12}$$

Where γ is surface tension and θ is contact angle.
Using equations (9)-(12), the equation of motion reduces to

$$\frac{1}{2}a^2z\frac{d^2z}{dt^2}=\frac{1}{\rho}\left[\frac{1}{2}a^2\left[P_0+\frac{2\gamma\cos\theta}{a}\right]-\int_0^a rp(0,t)dr\right]-4\nu z\frac{dz}{dt}+\frac{1}{2}ga^2z \tag{13}$$

Equation (13) can be simplified into equation (14):

$$z\frac{d^2z}{dt^2}+\frac{2\gamma\cos\theta}{\rho a}+\frac{8z\mu}{\rho a^2}\frac{dz}{dt}-gz=0 \tag{14}$$

Discussion

In steady state conditions, the acceleration of flow is negligible and we have

$$\frac{d^2z}{dt^2}=0 \tag{15}$$

$$\frac{dz}{dt}=U_m \tag{16}$$

Note that U_m is the average velocity because of definition of the poiseuille flow as equation (8).

Therefore,

$$\frac{2\gamma\cos\theta}{\rho a} + \frac{8z\mu}{\rho a^2}U_m - gz = 0$$

or

$$U_m = \frac{\rho g a^2 z - 2\gamma\cos\theta}{8\mu z} \tag{17}$$

and the volumetric flow rate is equal to Q

$$Q = \pi a^2 U_m = \frac{\pi a^3(\rho g a z - 2\gamma\cos\theta)}{8\mu z} \tag{18}$$

It is interesting to study the limiting cases as follow:

For very long capillary tube: In this case $z \to \infty$, therefore the equation (17) is reduced to

$$U_m = \frac{\rho g a^2}{8\mu} \tag{19}$$

Equation (19) is a general form of Darcy's law because

$$\frac{dP}{dx} = -\frac{\mu}{k}u \Rightarrow u = \frac{k}{\mu}\rho g$$

as $k \sim a^2$

so

$$u \sim \frac{\rho g a^2}{\mu} \tag{20}$$

hence, (19) and (20) have the same physical parameters.

For very short capillary tube: Equation (17) was derived by assuming steady state. When the length of capillary tube is very short, the end effects causes the violation of assumption, hence the length of capillary tube must satisfy the following inequality:

$$z > \frac{2\gamma\cos\theta}{\rho g a} \tag{21}$$

Equation (17) is a general form for low Reynolds number flow through porous media which includes all physical parameters, and represents Darcy's law for very long capillary tubes where the length of them satisfy the inequality (21).

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