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Mid-Point Euler Method in Pseudospectral Approximation for Burger's Equation

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Abstract: In this paper, a Fourier Pseudospectral Approximation combined with the Midpoint Euler time differencing technique for solving Burger's equation is proposed. The stability and convergence are investigated. The numerical results are also presented.

Key Words: Burger Equation, Fourier Pseudospectral Method, Stability, Convergence

Introduction
Consider the Burger's equation with periodic boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} + u u_x - \nu u_{x_x} = f, & -\infty < x < \infty, \quad o < t \le T, \\ u(x,t) = u(x+1,t), & -\infty < x < \infty, \quad 0 \le t \le T, \\ u(x,0) = u_0(x), & -\infty < x < \infty, \end{cases} \tag{1.1}$$

In the Past several years publications in spectral methods for nonlinear partial differential equation provide a new potent solution technique Canuto *et al.*, (1987); Gottlieb and Orszag (1977); Mady and Quartroni (1981); Guo Ben-Yu and Ma-Heping (1986); Ma-Heping and Guo Ben-YU. (1988); Guo Ben-Yu (1988); Kreiss and Oliger (1979); Song and Cai-Ping (1999); Ma and Gu (1986) and Rashid (1993;1994) In many of the relevant papers, Pseudospectral methods are used, because they are more efficient than spectral methods. But some times Pseudospectral methods have a nonlinear instability which causes the weakens the non-linearity of the solution. In order to eliminate these phenomena, filtering technique is used (Ma and Gu, 1986).

In this paper a restrain operator R_r is used to develop fully discrete Fourier Pseudospectral Approximation combined with midpoint Euler time differencing technique for Burger's equation. The fully discrete scheme is proposed together with iteration procedure for solving this implicit scheme. The stability and convergence are proved. The numerical results are presented.

Notations and Lemmas: Let $I=(0,1)\in R$.

$$(u,v) = \int_0^1 u(x)\overline{v(x)}d_x,$$

$$\|u\|^2 = (u,v), \qquad |u|_1 = \left\|\frac{\partial u}{\partial x}\right\|.$$

Let N be positive integer and n be any integer

$$v_N = span\left\{ e^{n\pi i x} \Big/ |n| \le N \right\}.$$

$\dot{V}_N$ is a real value function subset of V_N. Let P_N be the orthogonal projection operator form $L^2(I)$ to V_N, that is, $(P_N v, w)=(v, w), \quad \forall w\in V_N$.

let P_c be the interpolation operator form C(I) to V_N such that $p_c v(x_j) = v(x_j), \quad x_j = \frac{j}{2N+1}, \quad j=0,1,\ldots 2N.$

From Gottlieb and Orszag, (1977),

$$(v,w)_N = (P_c v, P_c w)_N = (P_c v, P_c w), \quad \forall v, w\in C(I). \tag{2.1}$$

where

$$(v,w)_N = \frac{1}{2N+1}\sum_{j=0}^{2N} v(x_j)\overline{w(x_j)}.$$

Let $L^p(I) = \left\{ v \,\middle|\, \|v\|_{L^p} = \left(\int_0^1 |v(x)|\right)^{1/2} < \infty \right\}.$

In particular, the inner product and the norm are $(\cdot,\cdot)$ and $\|\cdot\|$ respectively, for any positive constant β,let

$$|v|_\beta = \left\|\frac{\partial^\beta v}{\partial x^\beta}\right\|, \quad H^\beta(I) = \left\{ v \,\middle|\, \|v\|_\beta = \sum_{q=1}^{\beta} |v|_q < \infty \right\}.$$

$H^\beta(I)$ is the complex interpolation space between $H^{[\beta]}(I)$ and $H^{[\beta]+1}(I)$. Define

$$H_p^\beta(I) = \{ v \mid v\in H^\beta(I), \qquad v(x) = v(x+1) \},$$

We denote by $C^q(0,T:H_p^\beta(I))$ the space of abstract function with norm $\|v\|_{C^q(0,T:H_p^\beta(I))} = \max_{0\le x\le q} \max_{0\le t\le T} \left\|\frac{\partial^k v(t)}{\partial t^k}\right\|_\beta$.

The restrain operator R_r is proposed in Ma and Gu (1986), such that for $v(x) = \sum_{|l|\le N} a_n \exp(2\pi i l x)$

Then, $$R_r v(x) = \sum_{|l|<N}\left(1-\left(\frac{l}{N}\right)^r\right) a_l \exp(2\pi i l x), \qquad \forall v\in V_N.$$

Lemma 1: Guo and Ma-Heping (1986). If $0 \le \mu \le \beta$, and $v\in H_p^\beta(\Omega)$, then

$$\|P_N v - v\|_\mu \le cN^{\mu-\beta}|v|_\beta,$$

if $\beta > 1/2$, in addition, then

$$\|P_c v - v\|_\mu \le cN^{\mu-\beta}|v|_\beta.$$

Lemma 2: Ma and Gu (1986) If $0 \le \mu \le \beta$, and $v \in V_N$, then

$$\|v\|_\beta \le cN^{\beta-\mu}\|v\|_\mu .$$

Lemma 3: Guo and Ma-Heping (1986) If v w $\in V_N$, then

(i) $|v|_1^2 \le nN^2\|v\|^2$,

(ii) $\|vw\|^2 \le (2N+1)\|v\|^2\|w\|^2$.

Lemma 4: Gottlieb and Orszag (1977) Let $0 \le \mu \le \beta \le r$, $v \in V_N$, then

$$\|R_r v - v\|_\mu \le cN^{\mu-\beta}|v|_\beta,$$

Lemma 5: Ma and Gu (1986), If $v \in V_N$, $w \in H_p^{3/2+\gamma}(I), \gamma > 0$, then

$$\left|\left(R_r v * \frac{\partial v}{\partial x}, v\right)\right| \le c_\gamma v\|w\|_{3/2+\gamma}\|v\|^2,$$

$$\left|\left(v * R_r \frac{\partial v}{\partial x}, w\right)\right| \le c_\gamma v\|w\|_{3/2+\gamma}\|v\|^2,$$

where c_γ is positive constant depending only on γ.

Lemma 6: Guo and Ma-Heping (1986), If v, w $\in V_N$, $\gamma>0$, then

$$|((R_r J(R_r w, v), v)| \le c_\gamma \|w\|_{3/2+\gamma}\|v\|^2.$$

Lemma 7: Rashid (1994) Assume that

(i) ${}^k(x)$ and $\rho^k(x)$ are non-negative, non decreasing functions,

(ii) M is non-negative constant,

(iii) $E^0 \le \rho$ and for all $k\tau \le T$

$$E^k(x) \le \rho^k(x) + M\tau\sum_{\xi=0}^{K-1} E^\xi(x),$$

then for all $k\tau \le T$

$$E^k(x) \le \rho^k(x)\exp(Mk\tau).$$

The Scheme: Let τ be the mesh spacing of the variable t, $S_\tau = \left\{k \middle| 0 \le k \le \left[\frac{T}{\tau}\right] - 1\right\}$, and $v^k(x) = v(x, k\tau)$.

Define

$$v_t^k(x) = \frac{1}{\tau}\left[v^{k+1}(x) - v^k(x)\right],$$

$$\hat{v}^k(x) = \frac{1}{2}\left[v^{k+1}(x) + v^k(x)\right],$$

It is easy to see that,

$$2\left(\hat{v}^k(x), v_t^k(x)\right) = \left(\|v^k(x)\|^2\right)_t - \tau\|v_t^k(x)\|^2,$$

To approximate the non-linear term in (1.1), we define the operator J(. , .) as follows

$$J(u,v) = \frac{1}{3}u_x * R_r v + \frac{1}{3}(u * R_r v)_x,$$

A kind of Pseudospectral scheme with restraint operator R_r as the filter is to find $u^k(x) \in (V_N)^n$ for $k\tau \in S_\tau$, such that

$$\begin{cases} u_t^k(x) + R_r J\left(R_r\hat{u}^k(x), \hat{u}^k(x)\right) - v\nabla^2\hat{u}^k(x) = R_r P_c \hat{f}^k(x), \\ u^k(x) = u^k(x+1), \\ u^0(x) = P_c u_0, \end{cases} \tag{3.4}$$

The equation (3.4) is a nonlinear implicit scheme, a iteration procedure is proposed to solve the nonlinear implicit algebraic systems resulted from above scheme at each time level t,

$$\begin{cases} \dfrac{u^{k+1} - u(t)}{2} + R_r J\left(R_r\left(\dfrac{u^k + u(t)}{2}\right), \dfrac{u^k + u(t)}{2}\right) - v\nabla^2\dfrac{u^{k+1} + u(t)}{2} = R_r P_c \hat{f}(t), \quad k = 0,1,2,\dots \\ u^0 = u(t) \end{cases} \tag{3.5}$$

The Stability: Assume that $u^0(x)$ and $\hat{f}^k(x)$ have respectively the error $\tilde{u}^0(x)$, and $\tilde{\hat{f}}^k(x)$. Then the error of $u^k(x)$, denoted by $\tilde{u}^k(x)$ satisfies

$$\begin{cases} \tilde{u}_t^k(x) + R_r J\left(R_r\hat{u}^k(x), \hat{\tilde{u}}^k(x)\right) + R_r J\left(R_r\hat{\tilde{u}}^k(x), \hat{u}^k(x) + \hat{\tilde{u}}^k(x)\right) - v\nabla^2\hat{\tilde{u}}^k(x) = \tilde{\hat{f}}^k(x), \\ u^0(x) = \tilde{u}^0(x), \end{cases} \tag{4.1}$$

Let m>1 and $\epsilon > 0$, be constant chosen below. Taking the inner product of (4.1) with $2\hat{\tilde{u}}^k(x)$, we have

$$\left(\|\tilde{u}^k(x)\|^2\right)_t + +2v|\tilde{u}^k(x)|_1^2 + F \le c\left(\|\tilde{u}^k(x)\|^2 + \|\tilde{u}^{k+1}(x)\|^2\right) + G^k, \tag{4.2}$$

where

$$F = \left(R_r J(R_r\hat{u}^k(x), \hat{\tilde{u}}^k(x)), 2\hat{\tilde{u}}^k(x)\right),$$

$$G^k = c\left(\|\tilde{f}^k(x)\|^2 + \|\tilde{f}^{k+1}(x)\|^2\right)$$

we estimate $|F|$, by using Lemma 5 and lemma 6, we have

$$|F_1^k(x)| \le \varepsilon v\|\tilde{u}_t^k(x)\|^2 + \left(c + \frac{c}{\varepsilon v}\|u\|_{3/2+\varepsilon}\right)\left(\|\tilde{u}^k(x)\|^2 + \|\tilde{u}^{k+1}(x)\|^2\right)$$

By putting the previous estimation in to (4.2) , we get

$$\left(\|\tilde{u}^k(x)\|^2\right)_t + (2v - 3\varepsilon)|\tilde{u}^k(x)|_1^2 \le A\left(\|\tilde{u}^k(x)\|^2 + \|\tilde{u}^{k+1}(x)\|^2\right) + G^k(x), \tag{4.3}$$

where,

$$A = \left(c + \frac{c}{\varepsilon v}\|u\|_{3/2+\varepsilon}\right)\|u\|_{3/2+\varepsilon},$$

Let ε and τ be suitably small and define

$$E^k(x) = \|\tilde{u}^k(x)\|^2 + v\tau\sum_{\xi=0}^{k-1}|\hat{\tilde{u}}_t^\xi(x)|_1^2,$$

$$\rho^k(x) = \|\tilde{u}^0(x)\|^2 + v\tau\sum_{\xi=0}^{k-1} G^\xi$$

Then (4.3) leads to

$E^k(x) \le \rho^k(x) + 4A\tau\sum_{\xi=0}^{K-1} E^{\xi}(x)$,

by applying lemma (7) we finally reach the estimate regarding the generalized stability as follows.

Theorem 4.1: If $\tau = O(N^{-s})$, and $s \ge 2$. For certain $c > 0$, $\rho^k(x) \le c$. Then for all $k\tau \le T$, we have

$$E^k(x) \le \rho^k(x) e^{4Ak\tau}.$$

The Convergence: we next consider the convergence of the scheme (3.4) Let $w^k(x) = P_N u^k(x)$ and $e^k(x) = u^k(x) - w^k(x)$, then

$$\begin{cases} e_t^k(x) + J\left((\hat{e}^k(x), \hat{e}^k(x) + \hat{w}^k(x)\right) + J\left((\hat{w}^k(x), \hat{e}^k(x)\right) \\ -\nu\nabla^2 \hat{e}^k(x) = -\sum_{i=1}^{3} g_i^k, e^0(x) = (P_N - P_c)u_0, \end{cases} \quad (5.1)$$

where

$$g_1^k = \hat{f}^k(x) - R_r \hat{f}^k(x),$$

$$g_2^k = w_t^k(x) - \frac{\partial w}{\partial t}(x, k\tau),$$

$$g_3^k = R_r J\left(R_r \hat{w}^k(x), \hat{w}^k(x)\right) - P_N\left[u^k(x) u_x^k(x)\right].$$

By similar argument in the previous section we have

$$\left(\left\|e^k(x)\right\|^2\right)_t + (2\nu - 3\varepsilon)\left|\hat{e}^k(x)\right|_1^2 \le A\left(\left\|e^k(x)\right\|^2 + \left\|e^{k+1}(x)\right\|^2\right) + G_1^k(x), \quad (5.2)$$

where

$$A = \left(c + \frac{c}{\varepsilon\nu}\|w\|_{3/2+\varepsilon}\right)\|w\|_{3/2+\varepsilon},$$

$$G_1^k(x) = c\sum_{j=1}^{3}\left\|g_j^k\right\|^2$$

$$E_1^k(x) = \left\|e^k(x)\right\|^2 + \nu\tau\sum_{\xi=0}^{k-1}\left|\hat{e}^{\xi}(x)\right|_1^2,$$

$$\rho_1^k(x) = \left\|e^0(x)\right\|^2 + \nu\tau\sum_{\xi=0}^{k-1} G_1^{\xi}$$

Thus in order to get rate of convergence, we need only to estimate ρ_1^k. Assume $\mu > n/2$ and that u, u^0 and f are suitably smooth

$$\tau\sum_{\xi=1}^{k-1}\left\|g_1^{\xi}\right\|^2 \le cN^{-2\mu}\|f\|_{L^2(0,T;H^{\mu+1})},$$

$$\tau\sum_{\xi=1}^{k-1}\left\|g_2^{\xi}\right\|^2 \le c\tau^4\|u\|_{H^3(0,T;L^2)}^2,$$

$$\tau\sum_{\xi=1}^{k-1}\left\|g_3^{\xi}\right\|^2 \le cN^{-2\mu}\|u\|_{\mu+1}^2 + c\tau^4\left[\|u\|_{C^1(0,T;L^\infty)}^2 + \|u\|_{C(0,T;H^1)}^2\right],$$

$$\times\left[\|u\|_{H^2(0,T;H^1)}^2 + \|u\|_{H^2(0,T;L^\infty)}^2\right]$$

$$\left\|e^0(x)\right\|_1^2 \le cN^{-2\mu}\|u_0\|_{\mu+1}^2,$$

Consequently we have $\rho_1^k(x) = O(\tau^2 + N^{-2s})$

Theorem 5.1: Assume that $\tau = O(N^{-s})$, $s > 2$, and $\mu > n/2$ such that

$$u \in H^3(0,T;L^2) \cap H^2(0,T;L^\infty \cap H^1) \cap C^1(0,T;H^1)$$

and $f \in L^2(0,T;H^{\mu+1})$. If conditions of Theorem (4.1) are fulfilled then there exists a constant M_2 depending only on ν and the norm of u and f in space mentioned above. Such that

$$E_1^k \le M_2(\tau^4 + N^{-2s})$$

Numerical Results: For the numerical experiments, the function f(x, t) in the equation (1.1) is constructed in such a way that the solution of the problem is of the form

$$u(x,t) = 1 + \frac{a(t)}{2}\left\{-\tanh\left[\frac{a(t)(x-t)}{4\nu}\right] + 2(x-t)\right\},$$

where $a(t) = \dfrac{a_0(t)}{1 + a(0)t}$.

Let

$$E(t) = \max_{0 \le j \le 2N} \frac{\left|u(x_j,t) - u^k(x_j,t)\right|}{\left|u(x_j,t)\right|},$$

The calculation is carried out with N=4. The numerical results are tabulated in Table 1, 2, 3, and 4.

(i) The numerical results of scheme (3.4) are much better than that of scheme (Guo and Ma-Heping., 1986).
(ii) The value of r in the restrain operator must be suitably chosen. If r is too large the filtering technique will be weakened. If r is too small, the approximation accuracy will be lowered. The suitable value of r is between 5 and 10. But the best value of r is different in different cases, in this case the best value of r is 5 (Table 1 and Table 2).
(iii) If the viscosity is large, then R_r take little effect (Table 1 and 2)
(iv) If the viscosity is small, then the restrain operator greatly improve the accuracy (Table 3 and4)

Table 1: $N=4, \nu=0.1, \tau=0.01, a(0)=0.1$

r/ t	∞	10	5	1
0.5	1.3561E-4	1.0139E-4	1.0354E-4	1.0247E-4
1.0	1.5350E-4	1.5135E-4	1.5350E-4	1.5147E-4
1.5	2.0221E-4	2.0137E-4	2.0352E-4	2.0144E-4
2.0	2.5210E-4	2.5134E-4	2.5349E-4	2.5146E-4
2.5	3.0221E-4	3.0138E-4	3.0352E-4	3.0150E-4
3.0	3.5219E-4	3.5136E-4	3.5350E-4	3.5147E-4
3.5	4.0217E-4	4.0134E-4	4.0349E-4	4.0146E-4
4.0	4.5216E-4	4.5133E-4	4.5347E-4	4.5145E-4
4.5	5.0216E-4	5.0132E-4	5.0347E-4	5.0144E-4
5.0	5.5216E-4	5.5132E-4	5.5347E-4	5.5144E-4

Table 2: N=4, ν= 0.01, τ=0.01, a(0)=0.1

r/ t	∞	10	5	1
0.5	1.8740E-4	1.8526E-4	1.7596E-4	1.8574E-4
1.0	2.3731E-4	2.3516E-4	2.2586E-4	2.3564E-4
1.5	2.8727E-4	2.8513E-4	2.7583E-4	2.8561E-4
2.0	3.3719E-4	3.3504E-4	3.2574E-4	3.3552E-4
2.5	3.8716E-4	3.8502E-4	3.7472E-4	3.8550E-4
3.0	4.3709E-4	4.3494E-4	4.2564E-4	4.8542E-4
3.5	4.8708E-4	4.8493E-4	4.7563E-4	4.8541E-4
4.0	5.3701E-4	5.3486E-4	5.2556E-4	5.3534E-4
4.5	5.8695E-4	5.8480E-4	5.7550E-4	5.8528E-4
5.0	6.3689E-4	6.3474E-4	6.2544E-4	6.3522E-4

Table 3: N=4, ν=0.001, τ=0.01,a(0)=0

r/t	∞	10	5	1
0.5	5.4506E-4	5.1824E-4	3.4402E-4	5.3994E-4
1.0	6.2228E-4	5.9480E-4	3.2975E-4	6.1268E-4
1.5	5.9006E-4	5.6282E-4	3.7919E-4	5.8279E-4
2.0	6.0707E-4	5.1599E-4	4.2857E-4	5.3745E-4
2.5	6.5610E-4	5.0420E-4	4.7795E-4	5.4867E-4
3.0	7.0550E-4	5.5359E-4	5.2734E-4	5.9795E-4
3.5	7.5479E-4	6.0298E-4	5.7674E-4	6.4722E-4
4.0	8.0420E-4	6.5238E-4	6.2614E-4	6.9663E-4
4.5	8.5367E-4	7.0173E-4	6.7548E-4	7.4598E-4
5.0	9.0297E-4	7.5114E-4	7.2489E-4	7.9551E-4

Table 4: N=4, ν=.0001, τ=.01, a(0)=0.1

r/t	∞	10	5	1
0.5	8.9249E-4	7.7364E-4	3.4152E-4	7.4920E-4
1.0	8.6052E-4	6.6663E-4	3.6335E-4	6.8010E-4
1.5	9.1075E-4	7.0553E-4	3.8157E-4	7.4570E-4
2.0	9.2981E-4	7.1533E-4	4.2630E-4	7.5396E-4
2.5	9.4458E-4	7.2073E-4	4.7132E-4	7.5960E-4
3.0	9.6209E-4	7.5248E-4	5.1690E-4	7.6965E-4
3.5	9.8372E-4	7.6271E-4	5.6307E-4	7.8500E-4
4.0	1.0096E-3	7.8015E-4	6.0984E-4	8.0591E-4
4.5	1.0399E-3	8.0368E-4	6.5715E-4	8.3185E-4
5.0	1.0735E-3	8.3210E-4	7.0489E-4	8.6192E-4

References

Canuto, C., M. Y. Hussaini, A. Quartroni and T.A. Zang, 1987. Spectral Methods in Fluid Dynamics, Springer-Verlag, Berlin.

Gottlieb, D. and S. A. Orszag, 1977. Numerical Analysis of Spectral Methods, SIAM-CBMS, Philadephia.

Guo Ben-Yu and Ma•Heping, 1986. Fourier Pseudo Spectral method for Burger's equation, Northeastern Math. J., Vol. 2, 383-394.

Guo Ben-Yu, 1988. Difference Methods for Partial Differential Equations, Sci. Press, Beijing.

Kreiss, H. O. and J. Oliger, 1979. Stability of the Fourier Methods SIAM J Numer. Anal 16, 421-433.

Mady, Y. and A. Quartroni, 1981. Legendre and Chebyshev Spectral approximation of Burgers equation, Math, Vol. 37, 321-322.

Ma-Heping and Guo Ben-YU, 1988. The Chebyshev Spectral Method for Burger's Like Equations, J. of Computational Mathematics, Vol. 6, 48-53.

Ma He-ping and Gu Ben-Yu, 1986. The Fourier Pseudo-Spectral Method with restraint Operator for K.D.V. Equation, Vol. 65, 120-137.

Rashid. A., 1993. Mid Point Euler in Fourier Spectral method for fluid flow with low Mach number, J. of Shanghai Univ. of Sci. and Tech. Shanghai, China. Part(I), Vol. 16, No. 4, 418-426, Part(II), 1994.Vol. 17, No.1, 53-61.

Song Nian He and Cai-Ping Yang, 1999. Combined Legendre Spectral Finite Element Method for the two dimensional unsteady Navier Stokes Equations, J. of Computational Math., Vol.17, 394-508.