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On Finite Union of Prime Submodules

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Abstract: Let R be an associative ring with identity and M be a unital left R-module. A proper submodule N of M is called prime if whenever $r \in R$, $m \in M$ and $rRm \subset N$, then $m \in N$ or $rM \subset N$. In this paper we show the following two result i) The Prime avoidance theorem for unital left modules over noncommutative rings ii) Let S be an $m-system$ subset of a ring R and S^* be an $S-system$ subset of an R-module M. Let N be a submodule of M which is maximal in $M-S^*$. If the ideal $(N:M)$ is maximal in $R-S$, then N is a prime submodule of M.

Key Words: Prime Submodule, Union of Submodules, M-System

Introduction

All rings are associative with identity and all modules are unital left modules. Let R be a ring and M be a left $R-$module. A proper submodule N of M is called a prime submodule if whenever $r \in R$, $m \in M$ and $rRm \subset N$, then $m \in N$ or $rM \subset N$.

In (Chin Pi Lu, 1997), Chin-Pi Lu proved the Prime avoidance theorem for modules over commutative rings with identity. We extend her result to modules over noncommutative rings with identity. In (S-System Subsets of Modules), we define $S-$system subsets of modules where S is a $m-$system. We generalize some results for $m-$system subsets of R to $S-$system subsets of modules.

A Prime Avoidance Theorem For Modules Over Noncommutative Rings: Let $N_1, N_2, \ldots, N_n$ be submodules of an $R-$module M. We call a covering $N \subset N_1 \cup N_2 \cup \ldots \cup N_n$ efficient if N is not contained in the union of any $n-1$ of the submodules $N_1, N_2, N_3, \ldots, N_n$, i.e. if no N_k is superfluous. We say that $N = N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient union if none of the N_k may be excluded.

If N, N_1, N_2 are submodules of a module such that $N \subset N_1 \cup N_2$ then $N \subset N_1$ or $N \subset N_2$. Hence a covering of a submodule by two submodules is never efficient. Thus $N \subset N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient covering only when $n > 2$ or $n = 1$.

The next lemma is known for a module over commutative rings(see 1, lemma 2.1). It is easily checked that lemma is true for a module over noncommutative rings.

Lemma 1: Let $N = N_1 \cup N_2 \cup \ldots \cup N_n$ be efficient union of submodules of an $R-$module M for $n > 1$. Then $\bigcap_{j \neq k} N_j = \bigcap_{j=1}^{n} N_j$ for all k with 1≤k≤n.

Lemma 2: Let $N \subset N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient covering consisting of submodules of an $R-$module where n>1. If $(N_j : M) \not\subset (N_k : M)$ for every $j \neq k$, then no N_k for $k \in \{1, 2, \ldots, n\}$ is prime submodule of M.

Proof: By hypothesis, $N = (N \cap N_1) \cup (N \cap N_2) \cup \ldots \cup (N \cap N_n)$ is efficient union. Otherwise, for example, if $N = (N \cap N_2) \cup (N \cap N_3) \cup (N \cap N_4) \cup \ldots \cup (N \cap N_n)$,then $N \subset N_2 \cup N_3 \cup \ldots \cup N_n$. This is impossible. Since $N \subset N_1 \cup N_2 \cup \ldots \cup N_n$ is efficient covering $(n > 1)$, there exist an element a_k in $N - N_k$ for every k ≤ n. Morever, $\bigcap_{j \neq k} (N \cap N_j) = \bigcap_{j=1}^{n} (N \cap N_j)$ $\subset N \cap N_k$ by lemma 1. Suppose that N_k is a prime submodule. Then $(N_k : M)$ is a prime submodule of R. Since $(N_j : M) \not\subset (N_k : M)$ whenever $j \neq k$, then $(N_1 : M)(N_2 : M) \ldots (N_{k-1} : M)(N_{k+1} : M) \ldots (N_n : M) \not\subset (N_k : M)$.

There exist an element $r = \prod_{j \neq k} r_j \in (N_j : M)$ for every $j \neq k$ but $r \notin (N_k : M)$. Therefore, $rRa_k \subset N \cap N_j$ for every $j \neq k$ but $rRa_k \not\subset N \cap N_k$ which contradicts to

$\bigcap_{j\neq k}(N\cap N_j)=\bigcap_{j=1}^{n}(N\cap N_j) \subset N\cap N_k$. Consequently, no N_k is prime.

Theorem 3: Let M be an $R-$module, $N_1, N_2, N_3, ..., N_n$ a finite number of submodules of M and N be a submodule of M such that $N \subset N_1 \cup N_2 \cup ... \cup N_n$. Assume that at most two of the N_k's are not prime and that $(N_j : M) \not\subset (N_k : M)$ whenever $j \neq k$. Then $N \subset N_k$ for some k.

Proof: We may assume that covering is efficient without loss generality. Then $n \neq 2$. By lemma 2, $n \leq 2$. Then $n = 1$ and so $N \subset N_k$ for some k.

Proposition 4: Let M be an $R-$module, $N_1, N_2, ..., N_k$ a finite number of prime submodules of M and N be a submodule of M such that $N \subset N_1 \cup N_2 \cup ... \cup N_k$. If N is the radical of a cyclic submodules of M, then $N \subset N_i$ for some i.

Proof: Let $N \subset \bigcup_{i=1}^{n} N_i$ where $N_1, N_2, ..., N_n$ are prime submodules of M. By hypothesis $N = \sqrt{Rm}$ for some $m \in M$ and so $\sqrt{Rm} \subset \bigcup_{i=1}^{n} N_i$. Then $m \in N_i$ for some i. Since N_i is a prime submodule of M, $N = \sqrt{Rm} \subset N_i$ for some i.

S-System Subsets of Modules: A nonempty subset $S \subset R$ is called an $m-$system if, for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$. (see 2, for more detail)

Definition 5: Let M be an $R-$module and S be an $m-$system. A nonempty subset $L \subset M$ is called an $S-$system if, for every $a \in S$ and $m \in L$, there exist $r \in R$ such that $arm \in L$.

Proposition 6: Let M be an $R-$module and N be a prime submodule of M such that $(N : M) = P$ where P is a prime ideal of R. Then $M - N$ is an $S-$system where $S = R - P$.

Proof: Let N be a prime submodule of M and $(N : M) = P, S = R - P$. Let $a \in S$ and $t \in M - N$. Since N is a prime submodule of M, $aRt \not\subset N$. Therefore, there exist an $r \in R$ such that $art \in M - N$.

Proposition 7: Let $P_1, P_2, ..., P_n$ be a finite collection of prime submodules of M with $(P_j : M) = p_j$ and $(P_j : M) \not\subset (P_k : M)$, whenever $j \neq k$. Then $S^* = M - \bigcup_{i=1}^{n} P_i$ is a S-system subset of M, where $S = R - \bigcup_{i=1}^{n} p_i$.

Proof: Suppose $a \in R, m^* \in M$ and $arm^* \notin S^*$, for every $r \in R$. Then $aRm^* \subset \bigcup_{i=1}^{n} P_i$ where the P_i are prime submodules of M. Then $RaRm^* \subset \bigcup_{i=1}^{n} P_i$. Since $(P_j : M) \not\subset (P_k : M)$ whenever $j \neq k$, $RaRm \subset P_i$ by theorem 3 and so $aRm^* \subset P_i$ for some i. Then $m^* \in P_i$ or $a \in p_i$. As a result, $m^* \notin S^*$ or $a \notin S$.

Lemma 8: Let S^* be a $S-$system subset of an $R-$module M and N be a submodule contained in $M - S^*$. Then $(N : M) \cap S = \varnothing$.

Proof: Suppose that $(N : M) \cap S \neq \varnothing$ and $s \in (N : M) \cap S$. Then $sM \subset N$ and $sRf \subset N$ for every $f \in S^*$. This is impossible.

Theorem 9: Let S be a $m-$system subset of a ring R and S^* be an $S-$system subset of $R-$module M. Let N be a submodule of M which is maximal in $M - S^*$. If the ideal $(N : M)$ is maximal in $R - S$, then N is a prime submodule of M.

Proof: By lemma 8, $(N : M) \cap S = \varnothing$. Suppose that $m \notin N$ and $r \notin (N : M)$ for some $m \in M$ and $r \in R$ but $rRm \subset N$. Then, there exist $s^* \in M$ and $r^* \in R$ such that $s^* \in (N + Rm) \cap S^*$ and $r^* \in ((N : M) + RrR) \cap S$. Therefore, for every $t \in R$,

$r^*ts^* \in ((N : M) + RrR)R(N + Rm) = (N : M)RN + (N : M)Rm + RrRN + RrRm \subset N$.

This is impossible.

References

Chin Pi Lu; 1997. Unions of prime submodules, Houston J. of Mathematics, 23(2), 203-213.

Lam, T.Y.; 1999. Lecture on Modules and Rings, Springer –Verlag Newyork.