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# Cartesian Product on Fuzzy Prime Ideals

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**Abstract:** In this paper we examine the cartesian product of some fuzzy ideals. That is; if $\mu$ and $\sigma$ are fuzzy prime ideal, fuzzy semi prime ideal, fuzzy primary ideal, fuzzy semi primary ideals of $R$, then $\mu\times\sigma$ is fuzzy prime ideal , fuzzy semi prime ideal, fuzzy primary ideal, fuzzy semi primary ideals of $R\times R$. Moreover we showed that the level ideals of above fuzzy ideals of $R\times R$ are prime, semi prime, primary and semi primary ideals of $R\times R$.

**Key Words:** Fuzzy Ideal, Fuzzy Relation, Cartesian Product, Fuzzy Prime Ideal, Fuzzy Semi prime Ideal,Fuzzy Primary Ideal and Fuzzy Semiprimary Ideal

## Introduction

The concept of a fuzzy subset was introduced by Zadeh (1965). Fuzzy subgroup and its important properties were defined and established by Rosenfeld (1971). Then many authors have studied about it. After this time it was necessary to define fuzzy ideal of a ring. The notion of a fuzzy ideal of a ring was introduced by Liu. Malik, Mordeson and Mukherjee have studied fuzzy ideals (Malik and Mordeson, 1998). The concept of a fuzzy relation on a set was introduced by Zadeh (1965).

Bhattacharya and Mukherjee have studied fuzzy relation on groups. Malik and Mordeson studied fuzzy relation on rings (Malik and Mordeson, 1991). R. Kumar, V.N. Dixit and N. Ajmal studied on some definitions of fuzzy prime ideals (Kumar, 1992; Kumar *et al.*, 1992).

The aim of this paper is to analyse the cartesian products of some fuzzy prime ideals. In this paper $R$ is a commutative ring with identity. A fuzzy relation on $R$ is the fuzzy subset of $R\times R$. If $\mu$ and $\sigma$ are fuzzy prime ideals of $R$, then $\mu\times\sigma$ is fuzzy prime ideal of $R\times R$. Similarly if $\mu$ and $\sigma$ are fuzzy semi prime ideals of $R$, then $\mu\times\sigma$ is fuzzy semi prime ideal of $R\times R$. If $\mu$ and $\sigma$ are fuzzy primary ideals of $R$, then $\mu\times\sigma$ is fuzzy primary ideal of $R\times R$. If $\mu$ and $\sigma$ are fuzzy semi primary ideals of $R$, then $\mu\times\sigma$ is fuzzy semi primary ideal of $R\times R$. Moreover, we proved that if $\mu\times\sigma$ of $R\times R$ is fuzzy prime(primary) if and only if the level ideals $(\mu\times\sigma)_t$, $t\in \operatorname{Im}\mu\times\sigma$ are prime(primary) ideals of $R\times R$. Similarly a fuzzy ideal $\mu\times\sigma$ of $R\times R$ is fuzzy semi prime(semi primary) if and only if the level ideals $(\mu\times\sigma)_t$, $t\in \operatorname{Im}\mu\times\sigma$ are semi prime (semi primary) ideals of $R\times R$.

**Preliminaries:** In this section, we review some basic definitions and results

**Definition 1.1:** $A$ fuzzy subset of $R$ is a function $\mu : R \to [0,1]$.

**Definition 1.2:** $A$ fuzzy subset $\mu$ of $R$ is called a fuzzy left (right) ideal of $R$ if

(i) $\mu(x-y) \geq \min(\mu(x),\mu(y))$

(ii) $\mu(xy) \geq \mu(y)$ $(\mu(xy) \geq \mu(x))$ for all $x,y\in R$.

$A$ fuzzy subset $\mu$ of $R$ is called a fuzzy ideal of $R$ if $\mu$ is a fuzzy left and fuzzy right ideal of $R$.

**Definition 1.3:** If $\mu$ is a fuzzy subset of $R$, then for any $t\in \operatorname{Im}\mu$, the set $\mu_t = \{x\in R \mid \mu(x)\geq t\}$ is called the level subset of $R$ with respect to $\mu$.

**Theorem 1.4:** Let $\mu$ be fuzzy subset of $R$. $\mu$ is a fuzzy ideal of $R$ if and only if $\mu_t$ is an ideal of $R$ for $\forall t\in \operatorname{Im}\mu$.

Here, if $\mu$ is a fuzzy ideal of $R$, then $\mu_t$ is called a level ideal of $\mu$.

**Definition 1.5:** A fuzzy relation $\mu$ on $R$ is the fuzzy subset of $R\times R$.

**Definition 1.6:** Let $\mu$ and $\sigma$ be fuzzy subsets of $R$. The cartesian product of $\mu$ and $\sigma$ is $\mu\times\sigma(x,y)=\min(\mu(x),\sigma(y))$ for all $x,y\in R$.

**Definition 1.7:** A fuzzy ideal $\mu$ of $R$ is called fuzzy semi prime if $\mu(x^n)=\mu(x)$, $\forall x\in R$ and $\forall n \in {}^+$.

**Definition 1.8:** A fuzzy ideal $\mu$ of $R$ is called fuzzy semi primary if $\forall a,b\in R$ either $\mu(ab)\leq\mu(a^n)$ for some $n \in {}^+$ or else $\mu(ab)\leq\mu(b^m)$ for some $m\in {}^+$.

**Definition 1.9:** A fuzzy ideal $\mu$ of $R$ is called fuzzy primary if $\forall a,b\in R$ either $\mu(ab)=\mu(a)$ or else $\mu(ab)\leq\mu(b^m)$ for some $m\in {}^+$.

**Definition 1.10:** A fuzzy ideal $\mu$ of $R$ is called fuzzy prime if $\forall a,b \in R$ either $\mu(ab)=\mu(a)$ or else $\mu(ab)=\mu(b)$.

**Definition 1.11:** Let $\mu$ and $\sigma$ be fuzzy subsets of $R$. The cartesian product of $\mu$ and $\sigma$ is defined by $\mu\times\sigma=(x,y)=\min(\mu(x),\sigma(y)) \quad \forall x,y\in R.$

**Cartesian Products of fuzzy Prime Ideals**

**Theorem 2.1:** If $\mu$ and $\sigma$ be fuzzy prime ideals of $R$ then $\mu\times\sigma$ is a fuzzy prime ideal of $R\times R$.

**Proof:** We know that the cartesian product of any two fuzzy ideals is fuzzy ideal by (Malik and Mordeson, 1991). So it is enough to show that $\forall(a,b),(c,d)\in R\times R$ either

$$\mu\times\sigma((a,b)(c,d))=\mu\times\sigma((a,b))$$ or $$\mu\times\sigma((a,b)(c,d))=\mu\times\sigma((c,d))$$

Since $\mu$ and $\sigma$ is fuzzy prime ideals of $R$ $\forall a,b,c,d\in R$ either $\mu(ac)=\mu(a)$ or else $\mu(ac)=\mu(c)$ and either $\sigma(bd)=\mu(b)$ or else $\sigma(bd)=\mu(d)$. Then

$$\begin{aligned}\mu\times\sigma((a,b)(c,d)) &= \mu\times\sigma(ac,bd)\\ &=\min(\mu(ac),\sigma(bd))\\ &=\mu(ac)=\mu(a)=\mu\times\sigma(a,b) \text{ or}\\ &=\mu(ac)=\mu(c)=\mu\times\sigma(c,d).\end{aligned}$$

Similarly

$$\begin{aligned}\mu\times\sigma((a,b)(c,d)) &= \mu\times\sigma(ac,bd)\\ &=\min(\mu(ac),\sigma(bd))\\ &=\sigma(bd)=\sigma(b)=\mu\times\sigma(a,b) \text{ or}\\ &=\sigma(bd)=\sigma(d)=\mu\times\sigma(c,d).\end{aligned}$$

Therefore if $\mu$ and $\sigma$ are fuzzy prime ideals of $R$ then $\mu\times\sigma$ is a fuzzy prime ideal of $R\times R$.

**Corollary 2.2:** A fuzzy ideal $\mu\times\sigma$ of $R\times R$ is fuzzy prime if and only if the level ideals $t\in \mathrm{Im}(\mu\times\sigma)$, $(\mu\times\sigma)_t$ are prime ideals of $R\times R$.

**Proof:** $\Rightarrow$: We know that if $\mu\times\sigma$ is fuzzy ideal of $R\times R$ if and only if the level ideals $(\mu\times\sigma)_t$, $t\in\mathrm{Im}\,\mu\times\sigma$ are ideals of $R\times R$. So we must show that the level ideals are prime. $\forall(x,y),(z,t)\in(\mu\times\sigma)_t$ let $(x,y)(z,t)\in(\mu\times\sigma)_t$. Then

$$\begin{aligned}(x,y)(z,t)\in(\mu\times\sigma)_t &\Rightarrow (\mu\times\sigma)(xz,yt)\geq t\\ &\Rightarrow \min(\mu(xz),\sigma(yt))\geq t\\ &\Rightarrow xz\in\mu_t \text{ and } yt\in\sigma_t.\end{aligned}$$

Since $\mu_t$ and $\sigma_t$ are prime ideals of $R$, $(x,y),(z,t)\in\mu_t\times\sigma_t=(\mu\times\sigma)_t$. Therefor $t\in\mathrm{Im}(\mu\times\sigma)$, $(\mu\times\sigma)_t$ are prime ideals of $R\times R$.

$\Leftarrow$: It is clear that if $(\mu\times\sigma)_t$ is prime ideal then $\mu\times\sigma$ is fuzzy prime ideal of $R\times R$.

**Theorem 2.3:** If $\mu$ and $\sigma$ be fuzzy semiprime ideals of $R$ then $\mu\times\sigma$ is a fuzzy semiprime ideal of $R\times R$.

**Proof:** We know that the cartesian product of any two fuzzy ideals is fuzzy ideal by(Malik and Mordeson, 1991). So it is enough to show that $\forall(a,b)\in R\times R$ either $\mu\times\sigma((a,b)^n)=\mu\times\sigma(a,b)$. Since $\mu$ and $\sigma$ is fuzzy semi prime ideals of $R$ $\forall a,b\in R$, $\forall n\in{}^+$. $\mu(a)^n=\mu(a)$ and $\sigma(b)^n=\mu(b)$. Then

$$\begin{aligned}\mu\times\sigma((a,b)^n) &= \mu\times\sigma(a^n,b^n)\\ &=\min(\mu(a^n),\sigma(b^n))\\ &=\mu(a^n)\\ &=\mu(a)\\ &=\mu\times\sigma(a,b)\end{aligned}$$

or

$$\begin{aligned}\mu\times\sigma((a,b)^n) &= \mu\times\sigma(a^n,b^n)\\ &=\min(\mu(a^n),\sigma(b^n))\\ &=\sigma(b^n)\\ &=\sigma(b)\\ &=\mu\times\sigma(a,b).\end{aligned}$$

Therefore $\mu\times\sigma$ is a fuzzy semiprime ideal of $R\times R$.

**Corollary 2.4:** A fuzzy ideal $\mu\times\sigma$ of $R\times R$ is fuzzy semiprime ideal if and only if the level ideals $t\in\mathrm{Im}\,\mu\times\sigma$, $(\mu\times\sigma)_t$ are semiprime ideals of $R\times R$.

**Proof:** It is similar to proof of Corollary 2.2.

**Theorem 2.5:** If $\mu$ and $\sigma$ be fuzzy primary ideals of $R$ then $\mu\times\sigma$ is a fuzzy primary ideal of $R\times R$.

**Proof:** We know that the cartesian product of any two fuzzy ideals is fuzzy ideal by (Malik and Mordeson, 1991). Since $\mu$ and $\sigma$ be fuzzy primary ideals of $R$ then $\forall a,b,c,d\in R$ either $\mu(ab)=\mu(a)$ or else $\mu(ab)\leq\mu(b^m)$ for some $m\in{}^+$ and either $\sigma(cd)=\sigma(c)$ or else $\sigma(cd)\leq\sigma(d^n)$ for some $n\in{}^+$. Then we have the following choices.

i. If $\mu(ab)=\mu(a)$ and $\sigma(cd)=\sigma(c)$ then

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \mu(ab)\\ &= \mu(a)\\ &= \mu\times\sigma(a,c)\end{aligned}$$

or

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \sigma(cd)\\ &= \sigma(c)\\ &= \mu\times\sigma(a,c).\end{aligned}$$

ii. If $\mu(ab)\le\mu(b^m)$ and $\sigma(cd)\le\sigma(d^n)$ for some $m, n\in^+$ then

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \mu(ab)\\ &\le \mu(b^{\max(m,n)})\\ &= \mu\times\sigma(b,d)^{\max(m,n)}\end{aligned}$$

or

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \sigma(cd)\\ &\le \sigma(d^{\max(m,n)})\\ &= \mu\times\sigma(b,d)^{\max(m,n)}.\end{aligned}$$

Therefore $\mu\times\sigma$ is a fuzzy primary ideal of $R\times R$.

**Corollary 2.6:** A fuzzy ideal $\mu\times\sigma$ of $R\times R$ is fuzzy primary ideal if and only if the level ideals $t\in\operatorname{Im}\mu\times\sigma$, $(\mu\times\sigma)_t$ are primary ideals of $R\times R$.

**Theorem 2.7:** If $\mu$ and $\sigma$ be fuzzy semi primary ideals of $R$ then $\mu\times\sigma$ is a fuzzy semiprimary ideal of $R\times R$.

**Proof:** We know that the cartesian product of any two fuzzy ideals is fuzzy (Malik and Mordeson, 1991). Since $\mu$ and $\sigma$ be fuzzy semi primary ideals of $R$ then $\forall a,b,c,d\in R$ either $\mu(ab)\le\mu(a^n)$ or else $\mu(ab)\le\mu(b^m)$ for some $m$, $n$ $\in^+$ and either $\sigma(cd)\le\sigma(d^k)$ or else $\sigma(cd)\le\sigma(d^l)$ for some $k\ l\in^+$. Let $t=\max(m,n)$ and $s=\max(k,l)$. Then

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \mu(ab)\\ &\le \mu(a^t)\\ &= \mu\times\sigma((a,c)^t)\end{aligned}$$

or else

$$\begin{aligned}\mu\times\sigma((a,c)(b,d)) &= \mu\times\sigma(ab,cd)\\ &= \min(\mu(ab),\sigma(cd))\\ &= \sigma(cd)\\ &\le \sigma(d^s)\\ &= \mu\times\sigma((b,d)^s)\end{aligned}$$

Therefore $\mu\times\sigma$ is a fuzzy semi primary ideal of $R\times R$.

**Corollary 2.8:** A fuzzy ideal $\mu\times\sigma$ of $R\times R$ is fuzzy semi primary ideal if and only if the level ideals $t\in\operatorname{Im}\mu\times\sigma$, $(\mu\times\sigma)_t$ are semiprimary ideals of $R\times R$.

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