Pakistan Journal of Applied Sciences 2(11) : 1025-1028, 2002

Legendre Functions Direct Method for Solving Linear Differential Equations

H. Parsian
Department of Physics, University of Bu Ali sina , Hamadan, Iran

Abstract: A direct method for solving linear differential equation under initial values using Legendre function is presented. An operational matrix introduce for operator of differential equation and it reduce into a set of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key Words: Linear Differential Equation, Operational Matrix, Legendre Function

Introduction

Linear Differential Equations (LDE) have received in dealing with various problems of dynamic systems. An differential equation can not be solved exactly always and there is various numerical methods for solving it. The main characteristic of this technique is that it reduces differential equation to a set of algebraic equations, thus greatly simplifying the problem.The approach is based on converting the LDE into set of algebraic equation by choosing the answer function as the series of

$$\Phi(x)=\{\phi_\circ(x),\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x),\phi_5(x),...\}$$

and operational matrix L for differential operator L. The elements

$$\phi_\circ(x),\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x),\phi_5(x),...$$

are the basis function, orthogonal on a certain interval [a,b]. This technique is to be used for variational problems by various basis functions, to applications of Walsh Functions (Chen and Hsiao, 1975), block pulse functions (Hwang and Shih, 1985), Laguerre Functions (Hwang and Shih, 1983), Legendre Functions (Chang and Wang, 1983) and (Razzaghi and Yousefi, 2000), Chebyshev Functions (Horng and Chou, 1985) and Fourier series (Razzaghi and Razzaghi, 1988).

In the present paper, we introduce a direct computational method to solve LDE under initial values. This method consists of reducing the LDE into a set of algebraic equations by first expanding the candidate function as Legendre Functions with unknown coefficients. The operational matrix of differentation of $y(x)$, variable x, numbers and some of other function are given. The Legendre polynomials of order m are orthogonal with respect to the weight function $\omega(x)=1$ on the interval [-1,1]. Then the differential equation $Ly(x)=0$ reduce into $\mathrm{L}\Phi(x)=0$, in which L matrix operator of differential operator L and $0=[0,0,0,...,0]^T$ is the column matrix, and finally from solving $\mathrm{L}=0$, exact solution for LDE obtained.

Properties of Legendre Functions

Legendre functions: Legendre Polynomials is the solutions of DE below

$$(1-x^2)y''(x)-2xy'(x)+n(n+1)y(x)=\circ \qquad (1)$$

These are a set of orthogonal functions with respect to the weight function $\omega(x)=1$ on the interval [-1,1] and satisfy the following recursive formulas(Chen and Hsiao, 1997):

$$xP_n(x)=\left(\frac{n+1}{2n+1}\right)P_{n+1}(x)+\left(\frac{n}{2n+1}\right)P_{n-1}(x) \quad ,n=1,2,3,...$$
$$(1-x^2)P_n'(x)=nP_{n-1}(x)-nxP_n(x) \qquad (2)$$
$$P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$$

Function Approximation: A function $y(x)$ defined over [-1,1] may be expanded as

$$y(x)=\sum_{n=\circ}^{\infty}c_nP_n(x) \qquad (3)$$

where $c_n=\frac{2n+1}{2}<y(x),P_n(x)>$, in which $<\ ,\ >$ denotes the inner product.

If the infinite series in (3) is truncated, then (3) can be written as

$$y(x)=\sum_{n=\circ}^{N-1}c_nP_n(x) \qquad (4)$$

Matrices Definition

Column Matrices: If we definite C and $\mathrm{P}(x)$ as $N\times 1$ matrices below

$$\mathrm{C}=\left[c_\circ,c_1,c_2,c_3,c_4,...,c_{N-1}\right]^T$$
$$\mathrm{P}(x)=[P_\circ(x),P_1(x),P_2(x),P_3(x),P_4(x),...,P_{N-1}(x)]^T \qquad (5)$$

then (4) can be written as

$$y(x)=\sum_{n=\circ}^{N-1}c_nP_n(x)=\mathrm{C}^T.\mathrm{P}$$

The numbers and x^m can be defined as

$$1=[1,0,0,0,0,...,0,0]\mathrm{P}(x)=1^T.\mathrm{P} \qquad (6)$$
$$x=[0,1,0,0,0,...,0,0]\mathrm{P}=\mathrm{X}^T.\mathrm{P}$$
$$x^2=[\tfrac{1}{3},0,\tfrac{2}{3},0,0,...,0,0]\mathrm{P}=\mathrm{X}^{2^T}.\mathrm{P}(x)$$

$x^{2m} =$

$$\left[\frac{(2m)!}{m!(2m+1)!},0,\ldots,\frac{2^{2n}(4n+1)(2m)!(m+n)!}{(2m+2n+1)!(m-n)!},0,\ldots,0,\frac{2^{N-1}(2N-1)(2m)!(m+\frac{N-1}{2})!}{(2m+N)!(m-\frac{N-1}{2})!}\right]$$

$P(x) = X^{2m^T} . P_{(x)}$

$x^{2m+1} =$

$$\left[0,\frac{6(2m+1)!(m+1)}{(2m+3)!},0,\ldots,\frac{2^{2n+1}(4n+3)(2m+1)!(m+n+1)!}{(2m+2n+3)!(m-n)!},\ldots,\frac{2^{N-1}(2N-1)(2m+1)!(m+\frac{N}{2})!}{(2m+1+N)!(m-\frac{N-2}{2})!}\right] P$$

$(x) = X^{2m+1^T} . P_{(x)}$

Where 1, X, ... are $N\times 1$ corresponding operational matrices.

Differentation Matrix:

$$\left(\frac{d}{dx}\right)^m \equiv D^m$$

Differential of the function $y(x)$ defined in (4) can be obtained as

$$\frac{d}{dx}y(x) = \sum_{n=0}^{N-1} c_n P_n'(x) = \sum_{n=0}^{N-1} c_n$$

$$\left(P_{n-2}'(x) + (2n-1)P_{n-1}(x)\right)$$

$$= C^T.(U_{2n-1}.I^- + U_{2n-5}.[I^-]^3 + U_{2n-9}.[I^-]^5).P(x)$$

$$= C^T . D . P(x)$$

where U_{2n-1} and I^- are the $N\times N$ operational matrices that can be defined as

$$I^- = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

and

$$U_{2n-1} = \begin{bmatrix} -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \cdots & 2n-1 & \cdots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 2N-3 \end{bmatrix}$$

The first row and column of these matrices are $n=0$.

$$\frac{d^2}{dx^2}y(x) = \sum_{n=0}^{N-1} c_n P_n'(x) = \frac{d}{dx}\sum_{n=0}^{N-1} c_n$$

$$\left(P_{n-2}'(x) + (2n-1)P_{n-1}(x)\right)$$

$$= C^T.(U_{2n-1}.I^- + U_{2n-5}.[I^-]^3 + U_{2n-9}.[I^-]^5)^2.P$$

$$= C^T . D^2 . P(x)$$

finally

$$\frac{d^m}{dx^m}y(x) = \sum_{n=0}^{N-1} c_n P_n'(x) = \frac{d^{m-1}}{dx^{m-1}}\sum_{n=0}^{N-1} c_n$$

$$\left(P_{n-2}'(x) + (2n-1)P_{n-1}(x)\right)$$

$$= C^T.(U_{2n-1}.I^- + U_{2n-5}.[I^-]^3 + U_{2n-9}.[I^-]^5)^m . P_{(x)}$$

$$= C^T.D^m.P(x) \qquad (7)$$

Matrix: $x^m y \equiv C^T . X^m Y$

The term $xy(x)$ can be definite as

$$xy(x) = \sum_{n=0}^{N-1} c_n x P_n(x) = \sum_{n=0}^{N-1} c_n$$

$$\left(\frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)\right)$$

$$= C^T.(U_{\frac{n+1}{2n+1}} . I^+ + U_{\frac{n}{2n+1}} . I^-). P(x)$$

$$= C^T . XY. P(x) \qquad (8)$$

Where I^+ is the $N\times N$ operational matrix can be defined as

$$I^+ = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

The term $x^2 y(x)$ can be defined as

$$x^2 y(x) = \sum_{n=0}^{N-1} c_n x^2 P_n(x)$$

$$= C^T.\left(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^- \right)^2.P(x)$$

$$= C^T . X^2 Y. P(x) \qquad (9)$$

and also

$$x^3 y(x) = C^T.\left(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^- \right)^3.P(x)$$

$$= C^T . X^3 Y. P(x) \qquad (10)$$

finally

$$x^m y(x) = C^T.\left(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^- \right)^m.P(x)$$

$$= C^T . X^m Y. P \qquad (11)$$

Matrix: $x^m y' \equiv C^T . X^m Y'$

The term $xy'(x)$ can be definite as

$$xy'(x)=\sum_{n=0}^{N-1}c_n xP_n'(x)=\sum_{n=0}^{N-1}c_n$$
$$\left(xP_{n-2}'(x)+(2n-1)xP_{n-1}(x)\right)$$

$$xy'=C^T.XP'(x)$$
$$=C^T.U_{2n-1}.I^-.XP(x)+C^T.U_{2n-5}.[I^-]^3.$$
$$XP(x)+C^T.U_{2n-9}.[I^-]^5.XP(x)+...$$
$$=C^T.D.XY.P(x)$$
$$=C^T.XY'.P(x) \quad (12)$$

The term $x^2y'(x)$ can be definite as

$$x^2y'(x)=\sum_{n=0}^{N-1}c_n\left(x^2P_{n-2}'(x)+(2n-1)x^2P_{n-1}(x)\right)$$
$$=C^T.D.(XY)^2.P(x)$$
$$=C^T.X^2Y'.P(x) \quad (13)$$

and also

$$x^3y'(x)=\sum_{n=0}^{N-1}c_n\left(x^3P_{n-2}'(x)+(2n-1)x^3P_{n-1}(x)\right)$$
$$=C^T.D.(XY)^3.P(x)$$
$$=C^T.X^3Y'.P(x) \quad (14)$$

finally

$$x^my'(x)=\sum_{n=0}^{N-1}c_n\left(x^mP_{n-2}'(x)+(2n-1)x^mP_{n-1}(x)\right)$$
$$=C^T.D.(XY)^m.P(x)$$
$$=C^T.X^mY'.P(x) \quad (15)$$

Matrix: $x^my''\equiv C^T.X^mY''$

The term $xy''(x)$ can be definite as

$$xy''(x)=\sum_{n=0}^{N-1}c_n xP_n''(x)=\sum_{n=0}^{N-1}c_n$$
$$\left(xP_{n-2}''(x)+(2n-1)xP_{n-1}'(x)\right)$$
$$=C^T.XP''(x)=C^T.D^2.XP_{(x)}=C^T.D^2.XY.P(x)$$
$$=C^T.XY''.P(x) \quad (16)$$

The term $x^2y''(x)$ can be definite as

$$x^2y''(x)=\sum_{n=0}^{N-1}c_n x^2P_n''(x)$$
$$=\sum_{n=0}^{N-1}c_n\left(x^2P_{n-2}''(x)+(2n-1)x^2P_{n-1}'(x)\right)$$
$$=C^T.X^2P''(x)=C^T.D^2.(XY)^2.P(x)$$
$$=C^T.X^2Y''.P(x) \quad (17)$$

The term $x^my''(x)$ can be definite as

$$x^my''(x)=\sum_{n=0}^{N-1}c_n x^mP_n''(x)=\sum_{n=0}^{N-1}c_n$$
$$\left(x^mP_{n-2}''(x)+(2n-1)x^mP_{n-1}'(x)\right)$$
$$=C^T.D^2.(XY)^m.P(x)$$
$$=C^T.X^mY''.P(x) \quad (18)$$

and finally term $x^m\frac{d^{m'}}{dx^{m'}}y(x)$ can be definite as

$$x^m\frac{d^{m'}}{dx^{m'}}y(x)=\sum_{n=0}^{N-1}c_n$$
$$\left(x^m\frac{d^{m'}}{dx^{m'}}P_{n-2}(x)+(2n-1)x^m\frac{d^{m'-1}}{dx^{m'-1}}P_{n-1}(x)\right)$$
$$=C^T.D^n.(XY)^m.P(x) \quad (19)$$

Direct method of Legendre Functions: Consider the differential equation with the given initial values

$Ly(x)=0 \quad , \quad y(x_n)=y_n \qquad n=1,2,3,...,M$

where M is the order of differential equation. With choosing $y(x)=C^T.P_{(x)}$ and to find L operational matrix corresponding to L, we have

$$C^T.L.P(x)=0,\ C^T.P(x_n)=y_n \quad (20)$$

Where $0=[0,0,0,...,0]^T$ is a $N\times1$ matrix. With in mind that $P(x)$ is a set of orthogonal functions, then

$$C^T.L=0,\ C^T.P(x_n)=y_n$$

Illustrative Examples:

Example 1:

Consider the given LDE under initial value

$y'(x)+2y(x)=0 \quad , \quad y(0)=1$

Using Eq. (4)-(19), we get

Table 1: Estimated and Exact Values of $y(x)$

x	Estimated	Exact
-1.0	0.36787	0.36787
-0.8	0.44932	0.44932
-0.6	0.54881	0.54881
-0.4	0.67032	0.67032
-0.2	0.81873	0.81873
+0.0	1.00000	1.00000
+0.2	1.22140	1.22140
+0.4	1.49182	1.49182
+0.6	1.82212	1.82212
+0.8	2.22554	2.22554
+1.0	2.71828	2.71828

$$C^T.D+2.C^T=0,\quad C^T.P(0)=0 \quad (21)$$

With choosing $N=9$, and solving (21), The estimated and exact values of $y(x)$ are given in Table1.

Example 2:

Consider the given LDE under initial values

$y''(x) - y(x) = 0 \quad y(0) = 0, \quad y(1) = 1$

Using Eq. (4)-(19), we get

$C^T.D.D - C^T = 0, \quad C^T.P_{(0)} = 0, \quad C^T.P(1) = 1$ (22)

With choosing $N = 9$, and solying (22), The estimated and exact values of $y(x)$ are given in Table2.

Table 2: Estimated and Exact Values of $y(x)$

x	Estimated	Exact
+0.0	0.00000	0.00000
+0.1	0.08523	0.08523
+0.2	0.17132	0.17132
+0.3	0.25912	0.25912
+0.4	0.34951	0.34951
+0.5	0.44340	0.44340
+0.6	0.54174	0.54174
+0.7	0.64549	0.64549
+0.8	0.75570	0.75570
+0.9	0.87348	0.87348
+1.0	1.00000	1.00000

C= $[0,0.93910,0,0.59952,0,0.00093,0,0,0]^T$

The estimated solution is

$y(x) = 0.93910\, P_1(x) + 0.59952\, P_3(x) + 0.00093\, P_5(x) \quad 0 \le x \le 1$

exact solution is $y(x) = \dfrac{\sinh x}{\sinh 1}$

Example 3:

Consider the given LDE under initial value

$y'(x) - xy(x) = 0 \quad, y(0) = 2$

Using Eq. (4)-(19), we get

$C^T.D - C^T.XY = 0, C^T.P(0) = 2$ (23)

With choosing $N = 9$, and solving (23)

C= $[2.38991,0,0.83166,0,0.07206,0,0.00365,0,0.00013]^T$

The estimated solution is

$y(x) = 2.38991P_0(x) + 0.83166P_2(x) + 0.07206P_4(x) + 0.00365P_6(x) + 0.00013P_8(x)$

$-1 \le x \le 1$

and exact solution is $y(x) = 2e^{\frac{x^2}{2}}$.

Example 4:

Consider the given LDE under initial value

$y'(x) - x^2 y'(x) + xy(x) = 0, \quad y(0) = 1$

Using Eq. (4)-(19), we get

$C^T.D - C^T.X^2 Y' + C^T.XY = 0, C^T.P(0) = 1$ (24)

With choosing $N = 12$, and solving (24)

C=

$[0.78402,0,-0.49001,0,-0.11025,0,-0.04976,0,-0.02847,0,-0.01846,0]^T$

The estimated solution is

$y(x) = 0.78402P_0(x) - 0.49001P_2(x) - 0.11025P_4(x) - 0.04976P_6(x) - 0.02847P_8(x) - 0.01846P_{10}(x) \quad 0 \le x \le 1$

and exact solution is $y(x) = \sqrt{1 - x^2}$

Example 5:

Consider the given LDE under initial values

$$y^{(5)}(x) + y^{(4)}(x) - 360x^2 + 720 = 0$$
$$y(0) = 0, y(1) = -36, y(-1) = -23,$$
$$y(0.5) = -2.04688, \ y(-0.5) = -1.67188 \quad (25)$$

Using Eq. (4)-(19), we get

$C^T.[D]^5 + C^T.[D]^4 - 360X^2 + 720 = 0$ (26)

$C^T.P(0) = 0, C^T.P(1) = -36, C^T.P(-1) = -23, C^T.P(0.5) = -2.04688, C^T.P(-0.5) = -1.67188$

With choosing $N = 14$, and solving (26)

C= $[-5.85714,-2.57142,-16.66666,-2.66666,-6.54544,-0.76190,0.06926, -4.\times10^{-9},3.\times10^{-10},-1.\times10^{-11},9.\times10^{-13},-4.\times10^{-14},1.\times10^{-15},-7.\times10^{-17}]^T$

and exact solution is $y(x) = x^6 - 6x^5 - 30x^4$.

Conclusion

The operational matrix of differentition legendre function ,D, and other corresponding operators matrices and orthogonality of legendre functions, are used to solve LDE. The present method reduces a LDE into a set of algebraic equation and provide an exact solution almost. This method can be expanded for many LDE.

References

Chen, C.F. and C.H. Hsiao; 1975. AWalsh series direct method for solving variational problems, J. FranklinInstit.300, 265-280.

Chang, R.Y. and M.L. Wang; 1983. Shifted Legendre direct method for variational problems series, J. Optimization Theory App. 39, 299-307.

Chen, C. F. and C.H. Hsiao; 1997. Haar wavelet method for solving lumped and distributed-parameter Sys., IEE Proc. Control Theory Appl. 144, 87-93.

Hwang, C. and Y.P. Shih; 1983. Laguerre series direct method for variational problems, J. Optimization Theory Appl. 39,143-149.

Horng, I.R. and J.H. Chou; 1985. Shifted Chebyshev direct method for solving variational problems, Int. J. Sys. Sci. 16.855-861.

Hwang, C. and Y.P. Shih; 1985. Optimal control of delay systems via block pulse functions, J. Optimization Theory App. 45 ,101-112.

Razzaghi, M. and M. Razzaghi; 1988. Fourier series direct method for variational problems, Int. J. Control 48, 887- 895.

Razzaghi, M. and S. Yousefi; 2000. Legendre wavelets direct method for variational problems. Mathematics and Computers in Simulation 53 ,185-192.