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Cubic Macro-Element for Plates under Bending

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Abstract: Macro-elements are one of the powerful means in reducing number of equations to be solved in the finite element analysis. Especially when high order finite elements are used. This is because one Macro-element will represent many finite elements. In this paper a cubic macro-element for the analysis of plates under bending is developed. Implementation of the macro-elements in the analysis showed reduction in number of equations and excellent results were achieved. This new developed macro-element was tested and the results were compared with the results of conventional plate bending finite element solutions and with closed form solution if available.

Key Words: Macro-Elements, Plate Bending, Finite Elements

Introduction

Most structural systems inherently require numerous structural elements. The analysis of such systems using finite element method require large set of equations.

This is because of the necessity to use relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of Degrees of Freedom (d.o.f) needed to successfully model large systems.

The reduction of the total number of d.o.f will minimize the computer storage capacity and lower computational time and at the same time accurate results are achieved.

In this paper a cubic plate bending macro-element was developed. This Macro-Element (M.E) is based on transformation of many structural Finite Elements (F.E) into single equivalent macro-element.

This is done by preserving the same potential energies of the structure modeled by finite elements and the same structure modeled by macro-elements.

The finite elements inside a macro element are not necessary of the same order as that of the macro-element. For better convergence the order of the macro element must be at least of the same order of the finite elements inside the micro-element (Alani, H. R. Dynamic, 1983).

The developed macro-element is a cubic serendipity (C12) element Fig. 2.

This element is a plate bending element with (12) nodes and three d.o.f per node Fig. 1.

The displacement vector is (Harbock and Hrudey, 1984).

$$\lfloor u_i \rfloor = \lfloor u_{i1} \;\; u_{i2} \;\; u_{i3} \rfloor = [w_i \;\; w_{i,y} \;\; -w_{i,x}]$$

Where i = 1, 2, ... 12

Formulation of the Macro-Element: The macro-element stiffness matrix is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent macro-element model as follows:

$$V_o = V_m \qquad (1)$$

Where:

V_o: The strain energy of the original structure modeled by many finite elements that constitute one macro-element.

V_m: The strain energy of the macro-element.

$$\frac{1}{2} \lfloor u_o \rfloor [Sk_o] \{u_o\} = \frac{1}{2} \lfloor u_m \rfloor [K_m] \{u_m\} \qquad (2)$$

Where:

u_o: Displacement vector of the structure modeled by many finite elements that constitute one macro-element.

u_m: Displacement vector of one macro-element.

$[SK_o]$: The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element.

$[K_m]$: The stiffness matrix of the macro-element.

Let the displacement vector of the original structure, (which constitute one macro-element) $\{u_o\}$ be related to that of the macro-element $\{u_m\}$ as:

$$\{u_o\} = [T]\{u_m\} \qquad (3)$$

Where: [T] is the transformation matrix for the macro-element.

Substituting Eq. (3) into Eq. (2) gives:

$\lfloor u_m \rfloor [T]^T [SKo] [T] \{u_m\} = \lfloor u_m \rfloor [K_m] \{u_m\}$ therefor:

$$[T]^T [SK_o] [T] = [K_m] \qquad (4)$$

In the solution, matrix $[SK_o]$ is not needed, only $[K_o]$, the stiffness matrix of a single finite element bounded by the macro-element is needed. To explain this let.

n: The number of finite elements comprising the macro-element.

$[T_e]$: The finite-element transformation matrix.

Every time $[T_e]$ carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite-element under consideration. The transformed stiffness matrix for each finite-element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of $[K_m]$, as:

$$\sum_{e=1}^{n} [T_e]^T [K_o] [T_e] = [K_m] \qquad (5)$$

The transformation matrix $[T_e]$ is simply the evaluation of the shape functions of the macro-element at the nodes of the Finite-Element. This evaluation is based on local coordinates for the nodal points of the finite-elements with respect to the macro-element nodes.

The transformation matrix will depend on the Macro-Element type as follows:

Cubic C12 Serendipity Macro Element: The displacement functions over this element are expressed as follows:

$$W = \sum_{i=1}^{n} N_i W_i \; ; \; \theta_x = \sum_{i=1}^{n} N_i \, \theta_{xi} \;\; \& \;\; \theta_y = \sum_{i=1}^{n} N_i \, \theta_{yi}$$

Where i = 1,2,3......12

Ni: the shape function at node i

To construct [Te] consider Fig. 3. The transformation matrix [Te] of the Finite Element [k,L,m,n,o,p,q,r] which is inside the Macro-Element (1,2,3,4,5,6,7,8,9,10.11.12) will be as follows:

$$[Te] = \begin{bmatrix} T_{K1} & T_{K2} & T_{K3} & T_{K4} & T_{K5} & T_{K6} & T_{K7} & T_{K8} \ldots & T_{K12} \\ T_{L1} & T_{L2} & T_{L3} & T_{L4} & T_{L5} & T_{L6} & T_{L7} & T_{L8} \ldots & T_{L12} \\ T_{m1} & T_{m2} & T_{m3} & T_{m4} & T_{m5} & T_{m6} & T_{m7} & T_{m8} \ldots & T_{m12} \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & T_{n5} & T_{n6} & T_{n7} & T_{n8} \ldots & T_{n12} \\ T_{o1} & T_{o2} & T_{o3} & T_{o4} & T_{o5} & T_{o6} & T_{o7} & T_{o8} \ldots & T_{o12} \\ T_{p1} & T_{p2} & T_{p3} & T_{p4} & T_{p5} & T_{p6} & T_{p7} & T_{p8} \ldots & T_{p12} \\ T_{q1} & T_{q2} & T_{q3} & T_{q4} & T_{q5} & T_{q6} & T_{q7} & T_{q8} \ldots & T_{q12} \\ T_{r1} & T_{r2} & T_{r3} & T_{r4} & T_{r5} & T_{r6} & T_{r7} & T_{r8} \ldots & T_{r12} \end{bmatrix}$$

Where:

$$[T_{K1}] = \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_1 \end{bmatrix}$$

evaluated at node (K) of the F.E i.e. the participation of node [k] of the Finite Element that corresponds to node (1) of the macro-element under consideration.

In general:

$$[T_{ij}] = \begin{bmatrix} N_j & 0 & 0 \\ 0 & N_j & 0 \\ 0 & 0 & N_j \end{bmatrix}$$

Where: i = k , L , m , ... q , r the nodes of the Finite Element.

j = 1, 2 , 3 , ... 11,12 the nodes of the Macro Element.

Then:

$$\sum_{e=1}^{n} [T_e]^T_{36x24} [K_0]_{24x24} [T_e]_{24x36} = [K_m]_{36x36}$$

Macro-Element Load Vector: The externals loading are applied at known nodes of the Finite-Element model. However, these nodes may not necessarily coincide with the Macro-Elements nodes. It is required to calculate the equivalent consistent nodal load vector of each Macro-Element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original Finite Element nodes, as with the conventional Finite Element method. The nodal load vector of the F.E model can then be transformed to equivalent Macro-Element structural load vector by equating the external work done on the original structure modeled by Finite-Elements and that of the Macro-Element model as follows:

$$W_o = W_m \qquad (7)$$

Where:

W_o: The external work done on the F.E that constitute one Macro-Element.

W_m: The external work done on the macro-element.

$$\lfloor u_o \rfloor \{F_o\} = \lfloor u_m \rfloor \{F_m\} \qquad (8)$$

Where:

$\{F_o\}$: The assembled nodal load vector of the Finite-Elements constituting one Macro-Element.

$\{F_m\}$: The equivalent nodal load vector of the Macro-Element.

Substituting Eq. (3) into Eq. (8) gives:

$$\lfloor u_m \rfloor [T]^T \{F_o\} = \lfloor u_m \rfloor \{F_m\}$$

$$[T]^T \{F_o\} = \{F_m\} \qquad (9)$$

Where [T] is the same transformation matrix used in deriving $[K_m]$.

The Assembly of all the Macro-Element stiffness matrices into a structural stiffness matrix and also the construction of the Macro-Element structural load vector and solution of the structure equation are the same as that of conventional Finite Element method.

Applications: Two problems of plate bending analysis are solved and presented below in order to demonstrate the efficiency of the Macro-Elements developed.

The accuracy of the Macro-Elements are checked by using the conventional Finite Elements method and, if available, the exact solution.

Problem No. 1: The analysis of thin, square, isotropic cantilevers plate under combined concentrated and distributed loads, as shown in Fig. 4.

The following data are given for this problem:

L = 3m

T = 0.02 m

$E = 200*10^6$ KN/m^2

$G_{xy} = G_{xz} = G_{yz} = 76.9823 * 10^6$

Nu = 0.3

Loading Q_z:

On finite elements 1 to 6 : 1.0 KN/m^2

= = = 7 to 12 : 2.0 KN/m^2

= = = 13 to 18: 3.0 KN/m^2

The plate loading with Q_z is symmetrical with respect to the Y- axis.

Table 1: Details for Problem No. 1

Mesh	No of Nodes	Total d.o.f	Reduction Percent
6*6 (L4) conventional F.E	49	147	–
Equivalent M.E			
2*2 C12 M.E	33	99	32.6%
1*1 C12 M.E	12	36	75.5%
3*3 Q8 M.E	40	120	18.4%
2*2 Q8 M.E	21	63	57.1%

Table 2: Deflections at Point (A) and their Corresponding Errors for Problem 1

Mesh	Deflection (mm)	Error %
6*6 L4 conventional F.E	86.49	–
2*2 C12 M.E	84.67	2.10
1*1 C12 M.E	83.22	3.78
3*3 Q8 M.E	85.91	0.67
2*2 Q8 M.E	77.80	10.05

Loading P_z: Each finite element node along Y-axis is loaded with 0.25 KN.

The analysis is first done using (L4) finite elements and then (Q8 and C12) macro-elements are used.
The (Q8) macro-element (Alani and Nasser, 2001) is used to compare its results with (C12) macro-element.
Table (1) shows the details of the original F.E model and the equivalent M. E models.
The results for deflections along Y-axis and section A-A are shown in Figs (5 and 6). The anticlastic curvature is observed.
Table 2 shows the errors percent of deflections at point A of the plate when using the M.E (Q8 and C12). Values of errors are measured from the (6*6 L4) conventional F.E analysis for the problem.
Problem No. 2: The analysis of thin, square, simply supported isotropic plate under a uniformly distributed load, as shown in Fig. 7.

The following data are given for this problem:

$L = 10$ in units of length.
$T = 0.1$ in units of length.
$E = 10.92 * 10^7$ in units of force/area.
$G_{xy} = G_{xz} = G_{yz} = 4.2 * 10^7$
$Nu = 0.3$
$Q_z = 1.0$ In units of force/area.

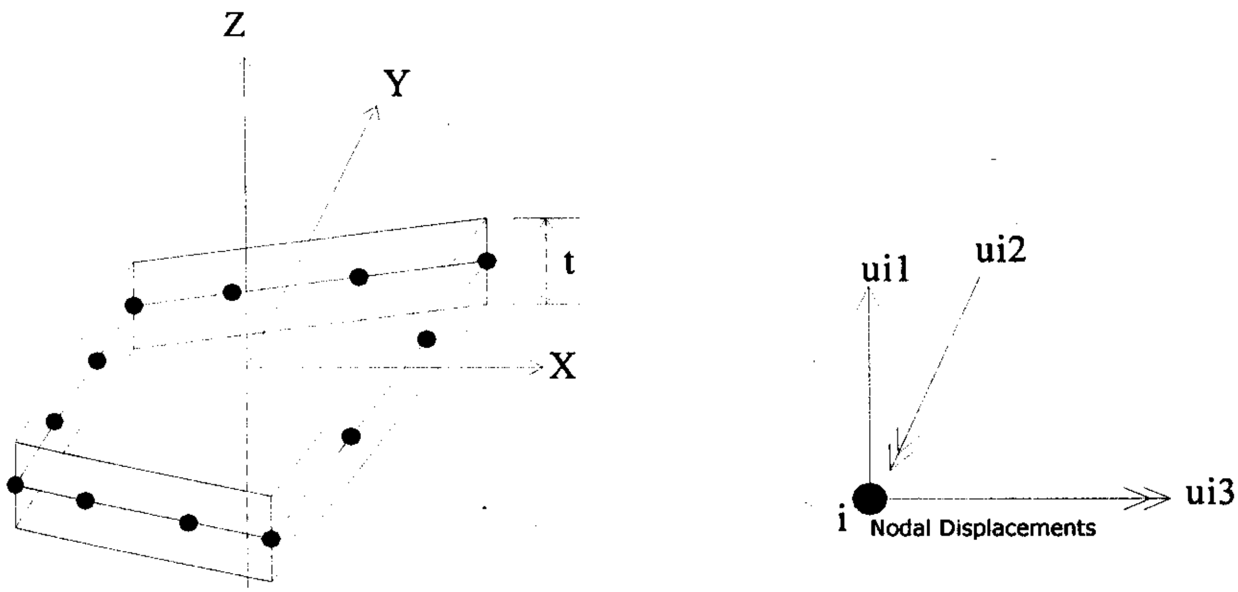
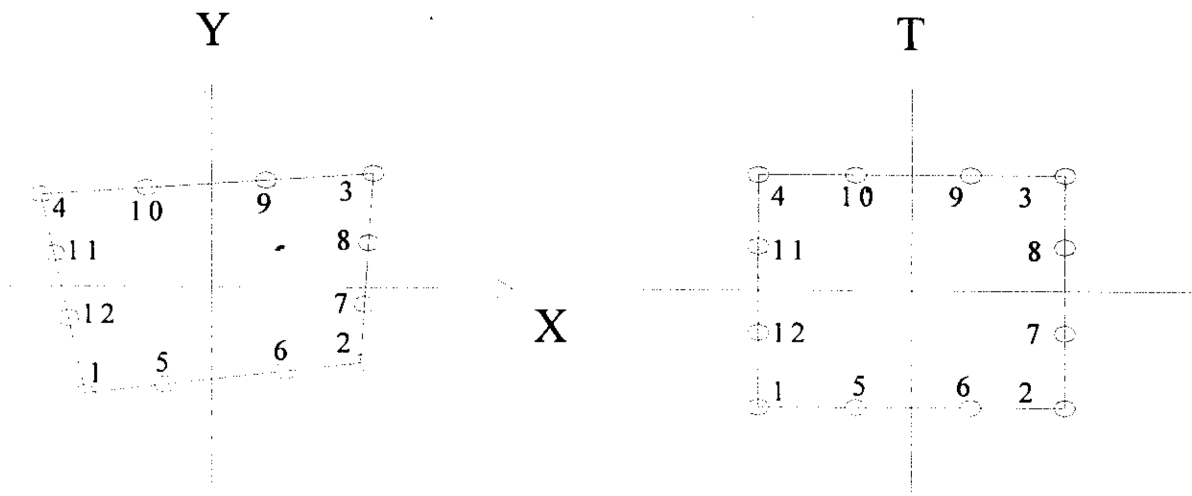


Fig. 1: A General Cubic Isoparametric Finite Element



Xy-Coordinate System St-Coordinate System

Fig. 2: The Cubic Serendipity (C12) Quadrilateral Isoparametric Finite Element

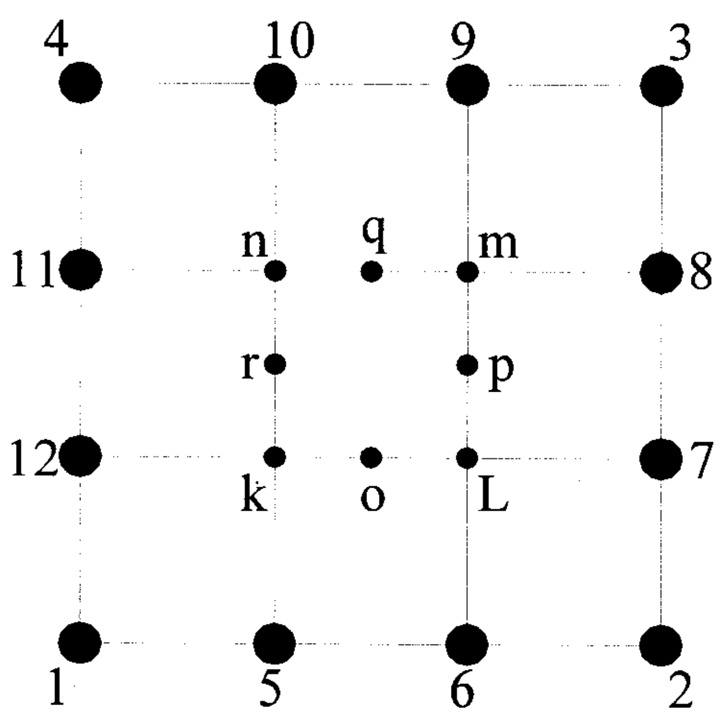


Fig. 3: The Correspondence Between the Finite Element DOF and the Macro Element DOF

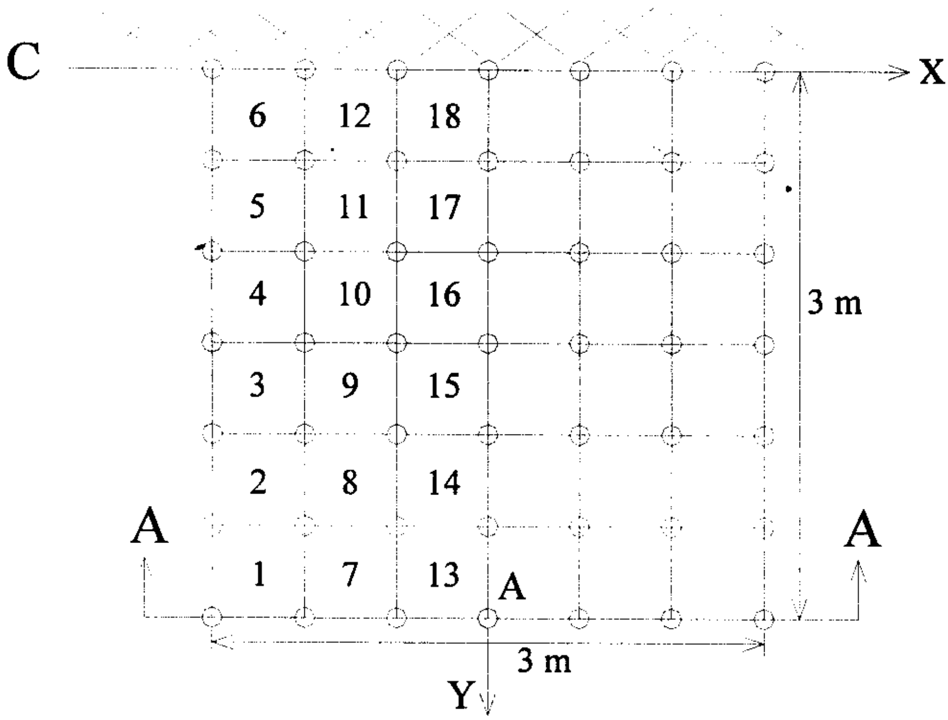


Fig. 4: Finite Discretization for the Cantilever Plate of Problem No.1 (Using 6*6 (L4) F.E.Mesh)

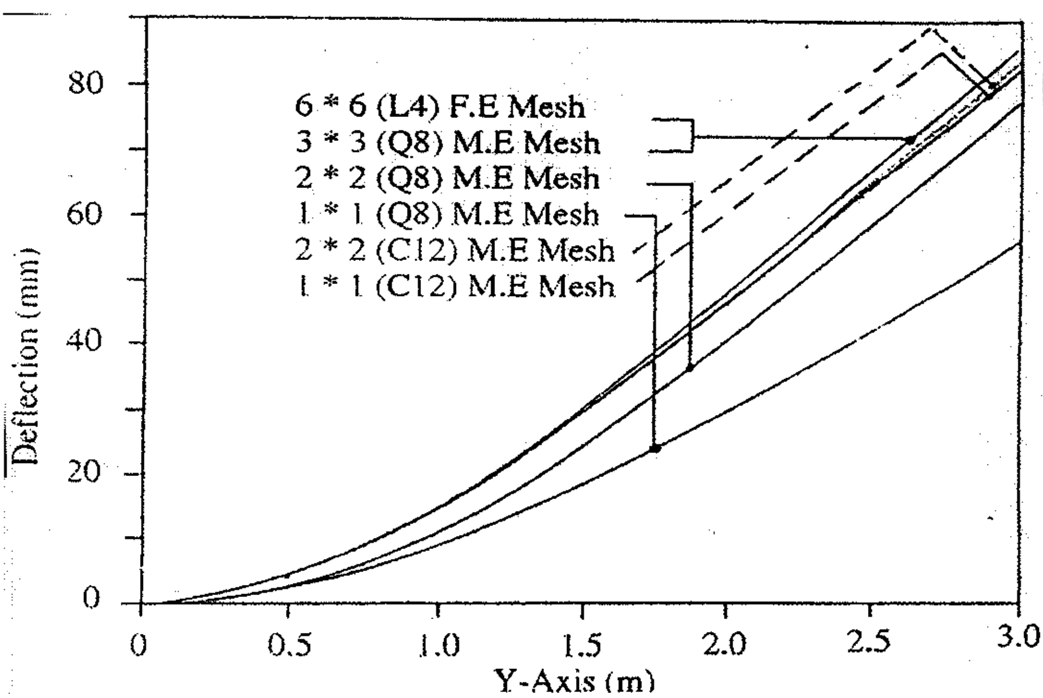


Fig. 5:Y-Axis Deflection for Problem No.1

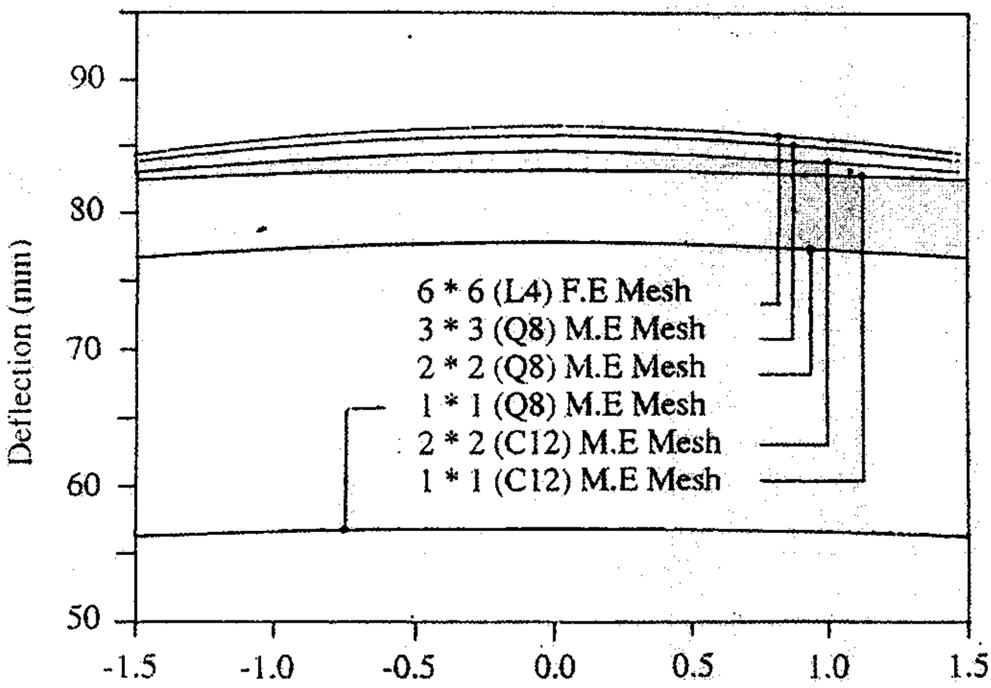


Fig. 6: Deflection Along Sec.A-A for Problem No.1

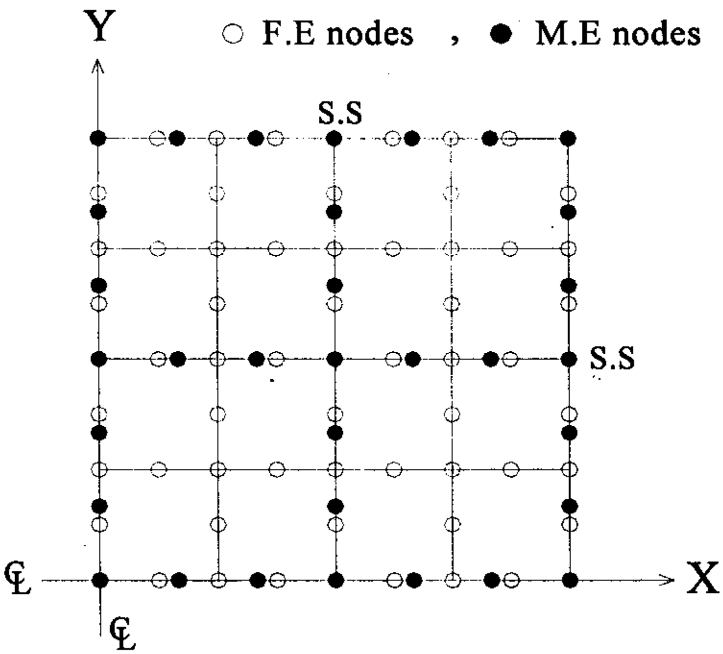


Fig. 7: Quarter of Plate for Problem No.2 Analyzed with the (Q8) F.E and C12 M.E.

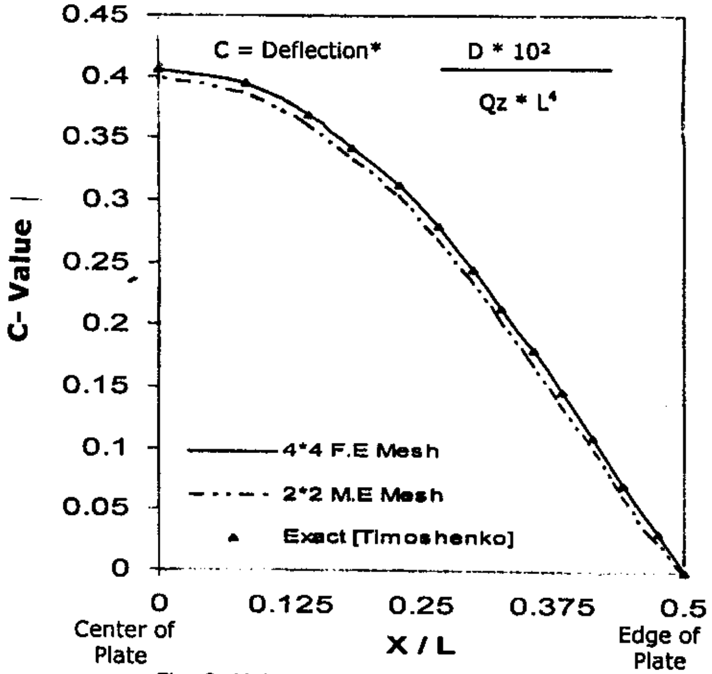


Fig. 8: X-Axis Deflection for Problem No.2

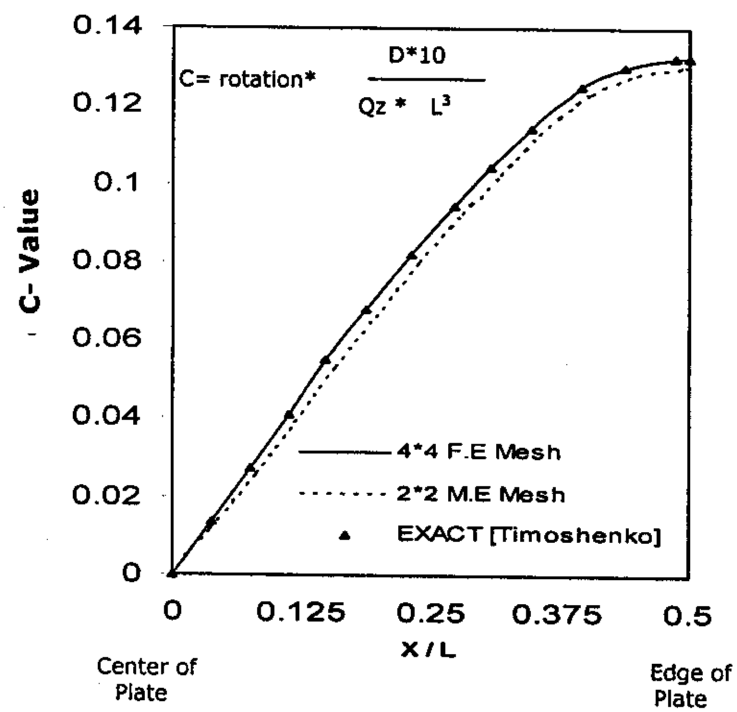


Fig. 9: X-Axis Rotation for Problem No.2

Table 3: A Comparative Study of Different (Q8) Meshes for Problem No. 1

Original F.E mesh	M.E Mesh	M.E size (F.E*F.E)	CPU (seconds)	C=central def $\frac{D * 10^2}{*Qz * L^4}$	Error % in defl.
(1)	(2)	(3)	(4)	(5)	(6)
	Closed form solution (Timoshenko)			0.4062	_
	conventional F.E. analysis		1287.8	0.4064452	0.0604
12*12 Q8	6*6	2*2	773.3	0.4069097	0.1740
	4*4	3*3	689.4	0.4075727	0.3379
	3*3	4*4	637.0	0.4086242	0.5968
	2*2	6*6	662.7	0.4158764	2.3822
	1*1	12*12	659.8	0.4218495	3.852

The results may be expressed in a normalized form as follows:

$$\text{Deflection} = C * Q_z * L^4 * 10^{-2} / D$$

$$\text{Rotations (in x or y)} = C * Q_z * L^3 * 10^{-1} / D$$

The aim of this problem is to see how the solution is effected when the size of the M.E is increased i.e when number of Finite Elements included in one M.E is increased and also to compare with the exact solution.

Due to symmetry only one quarter of plate is analyzed. The analysis is done using the (Q8) F.E, as shown in Fig. 7.

The original F.E mesh has 4*4 F.E with a total number of (65) nodes and (195) d.o.f. The equivalent M.E model has 2*2 M.E with a total number of (33) nodes and (99) d.o.f. The total reduction in d.o.f. is 50.77%.

The results for deflections and rotations are shown in Figs. (8 and 9). The maximum errors are (1.65%) and (0.88%) respectively.

Table (3) shows a comparative study for the execution time (CPU), the central deflections and their corresponding errors. The analysis is done using (Q8) conventional F.E. and (C12) equivalent M.E. meshes.

Results and Discussion

The two solved problems showed that using the Macro-Elements in the analysis reduced the number of equations to be solved. From problem 1 one can see that a reduction of 75.5% in d.o.f the error was only 3.78%.

Table 3 of problem 2 told us that when 12*12 F.E mesh with total of (2379) d.o.f was modeled by deferent M.E. meshes with total d.o.f ranging between (651 to 36) the error was ranging between (0.174% to 3.852%) which are acceptable.

This gives an idea that a moderate size of Macro Element used in the analysis will give acceptable error.

Conclusion

A new cubic plate bending Macro-Element (C12) is developed.

The solved examples demonstrated that using these Macro-Elements in the analysis largely reduced the total number of d.o.f required to model a certain structure. This in turn reduced the total number of equations to be solved.

Reduction in total number of equations reduced computer time and memory space for storage. And at the same time these M.E. provided accurate results. In addition, Finite Elements of different sizes, thicknesses and material properties can easily be used inside the Macro-Elements if required in the analysis.

References

Alani, H. R., 1983. Dynamic and Static Analysis of Large Structures Using Energy Equivalent Macro-Elements. Ph.D. Thesis Uni. of Colorado.

Alani, H. R. and I. B. Nasser, 2001 "Dev. of Quadratic Plate Bending Macro-Elements" Pak. J. of App. Sci., Vol. 1, No. 3, 417-425.

Harbock, M. M. and T.M. Hrudey, 1984. "A review and catalogue of plate bending finite elements" computer and structures, Vol. 19, No. 3, 479-495.

Reddy, J.N., 1985." An introduction to finite element method" McGraw-Hill.

Timoshenko, S. and S. Woinowsky- Krieger,1959. "Theory of plates and shills", 2nd Ed: McGraw-Hill, Inc.,Tokyo.