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# Convergence of Pseudospectral Method for Solving Navier-Stokes Equations

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**Abstract:** In this paper a new kind of Pseudospectral scheme is constructed for the Unsteady Navier-Stokes equations. This method easily deal with nonlinear terms and saves computational time. The generalized stability of the scheme is analyzed and the convergence is proved. Numerical results are presented also.

**Key Words:** Navier-Stokes Equations, Pseudospectral Method, Generalized Stability, Convergence

## Introduction

Consider the periodic problem of Navier Stokes equations as follows:

$$\begin{cases} \frac{\partial U}{\partial t}(x,t)+(U(x,t)\cdot\nabla)U(x,t)-.\ \nabla^2 U(x,t)+\nabla P(x,t)=f(x,t), \\ \nabla\cdot U(x,t)=0, \qquad (x,t)\in\Omega\times(0,T], \\ U(x,0)=U_0(x), \quad P(x,0)=P_0(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,2\pi)^n$, n = 2, or 3, $\nu \geq 0$, U = $(U_1,U_2,\ldots,U_n)$ is the velocity, P is the ratio of pressure to density. The functions $U_0, P_0$ and f are given with period 2π for all the space variables. We require in addition

$$\int_\Omega p(x,t)dx = 0 \qquad \forall t\in(0,T].$$

There is much literature concerning numerical solutions of Navier Stokes equation can be found in (Canuto *et al.*, 1988; Cao Weiming and Guo Benyu, 1991; Guo Benyu, 1985; Temam, 1977). Specific algorithm in (Huang Wei and Guo Ben-Yu, 1992; Jing-Yu Hou and Ben-Yu Guo, 1999; Ben-Yu Guo, 1996; Guo Ben-yu and Cao Weiming, 1992; Li Jian and Guo Ben-yu, 1995) have been devoted to the semi-periodic cases, which describes channel flow, parallel boundary layers curved channel flow and cylindrical couette flow. This paper is devoted to Fourier Pseudospeetral method for two-dimensional unsteady Navier stokes equations with periodic boundary conditions. This method is performed easily and has the same high accuracy as spectral method.

**Notations and Lemmas:** Denote by (.,.) and $\|\cdot\|$ the inner product and norm of $L^2(\Omega)$ respectively. Let $\|\cdot\|_\infty$ be the norm of $L^\infty(\Omega)$. Define $C_p^\infty(\Omega)=\{u|u\in C^\infty(\Omega)$, u has the period 2π for the variable $x_q, 1\leq q\leq n$ } and let $H_q^\mu(\Omega)$ be the closure of $C_p^\infty(\Omega)\ in\ H^\mu(\Omega)$.

Let x be a Banach space. Define

$$L^2(0,T;x)=\left\{u|u:[o,T]\to X, \|u\|_{L^2(0,T;X)}=\left(\int_0^T\|u(t)\|_X^2\,dt)<\infty\right)\right\},$$

$$C(0,T;X)=\left\{\begin{matrix} u|u;[o,T]\to X & \text{is strongly} \\ \text{continous} & \|\|u\|\|_X=\max_{0\leq t\leq T}\|u(t)\|_X \end{matrix}\right\}$$

Denote by z the set of integers. For $k=(k_1,k_2,\ldots,k_n)\in z^n$, let $|k|_\infty=\max_{1\leq q\leq n}|k_q|$, and $|k|=\left(\sum_{q=1}^n k_q^2\right)^{1/2}$. For positive integer N, we define

$$V_N=span\{e^{ik.x}|k\in z^n, |k|_\infty\leq N\},$$
$$W_N=span\{e^{ik.x}|x\in z^n, |k|\ \leq N\},$$

Let $P_N$ be the orthogonal projection operation form $L^2(\Omega)$ onto $W_N$. $\widetilde{P}_c$ is the Lagrange interpolation operator from $C(\Omega)$ onto $V_N$ at points $x^{(j)}=\frac{2\pi j}{(2N+1)}$, j $\in Z^n$, $|j|_\infty\leq N$. Let $P_c=P_N\widetilde{P}_c$. It is easy to see that $(P_c(u\ v), w)=(P_c(w\ v), u)$, $\forall u, v, w\in W_N$

**Lemma 2.1:** Rashid *et al.*, (1994) If u, v ∈ $V_N$, then

(i) $|u|_1^2\leq nN^2\|u\|^2$

(ii) $|uv|_1^2\leq n(2N+1)^n\left(\|u\|^2|v|_1^2+\|v\|\|v|_1^2\right)$

**Lemma 2.2:** Canuto and Quartroni (1982) If $0\leq\mu\leq\sigma$ and $u\in H_p^\sigma(\Omega)$ then

$$\|P_N u-u\|_\mu \ \leq\ CN^{\mu-\sigma}|u|_\sigma,$$

and if σ > n/2, then

$$\|P_c u-u\|_\mu \ \leq\ CN^{\mu-\sigma}|\eta|_\sigma.$$

**Lemma 2.3:** Canuto and Quartroni (1982) If $0 \le \mu \le \sigma$, and $u \in V_N$, then

$$\|u\|_\mu \le CN^{\mu-\sigma}\|u\|_\sigma.$$

**Lemma 2.4:** Song-nian and Cai ping (1999) Assume that following conditions are fulfilled

(i) E(t) is a non negative function defined on $R_\tau$,
(ii) $\rho, M_1$ and $M_2$ are non-negative constants;
(iii) for all $t \in R_\tau$,
(iv)

$$E(t) \le \rho + M_1\tau\sum_{t'=\tau}^{t-\tau}\left[E(t') + M_2E^2(t')\right],$$

(iv) $E(0) \le \rho$ and for some $t_1 \in R_\tau$,
$\rho e^{2M_1 t_1} \le \frac{1}{M_2}$, Then for all $t \in R_\tau$, $t \le t_1$, we have
$E(t) \le \rho e^{2M_1 t}$.

**The Pseudospectral Scheme:** Let $\tau$ be the mesh size of the variable t and

$$R_\tau = \left\{t \middle| t = k\tau, 1 \le k \le \left[\frac{T}{\tau}\right]\right\},$$

we denote u(x, t) by u(t) or u sometime. Let

$$u_{\hat{t}}(t) = \frac{1}{2\tau}\left[u(t+\tau) - u(t-\tau)\right],$$
$$\hat{u}(t) = \frac{1}{2}\left[u(t+\tau) + u(t-\tau)\right],$$

we use small parameter technique for continuity equation (Jing-Yu and Ben-Yu, 1999). Then the incompressible condition is approximated by

$$\beta\frac{\partial P}{\partial t} + \nabla\cdot U = 0.0, \qquad \beta > 0.$$

To approximate the nonlinear term , we define

$$d_c(u,v) = \sum_{j=1}^{2}\partial_j P_c(v^{(j)}u),$$

where $v^{(j)}$ is the component of v.
The Pseudospectral scheme for solving (1.1) is to find $u(t) \in V_N$ and $p(t) \in W_N$, for $t \in R_\tau$, such that

$$\begin{cases} u_{\hat{t}}(t) + d_c(u(t),u(t)) - \nu\nabla^2\hat{u}(t) + \nabla\hat{p}(t) = P_c f(t), \\ \beta p_{\hat{t}}(t) + \nabla\cdot\hat{u}(t) = 0.0, \\ u(0) = P_c U_0, \qquad u(\tau) = P_c(U_0 + \tau\partial_t U(0)), \\ p(0) = P_c P_0, \qquad p(\tau) = P_c(P_0 + \tau\partial_t P(0)), \end{cases} \tag{3.1}$$

**Stability:** Suppose that the initial values u(0), p(0) in (3.1) have errors $\tilde{u}_0$, $\tilde{p}_0$ and the right hand terms in the first and second equations have errors $\tilde{f}$ and $\tilde{g}$ respectively. Then the errors $\tilde{u}(t), \tilde{p}(t)$ of u(t) and p(t) satisfy

$$\begin{cases} \tilde{u}_{\hat{t}}(t) + d_c(u(t),\tilde{u}(t)) + d_c(\tilde{u}(t),u(t)+\tilde{u}(t)) - \nu\nabla^2\hat{\tilde{u}}(t) \\ + \nabla\hat{\tilde{p}}(t) = P_c\tilde{f}(t), \beta\tilde{p}_{\hat{t}}(t) + \nabla\cdot\hat{\tilde{u}}(t) = P_c\tilde{g}(t). \end{cases} \tag{4.1}$$

By taking the inner product of the first equation of (4.1) with $2\hat{\tilde{u}}(t)$ we get

$$\left(\|\tilde{u}(t)\|^2\right)_{\hat{t}} + 2\nu\left|\hat{\tilde{u}}\right|_1^2 + \sum_{j=1}^{2}F_j + 2\left(\nabla\hat{\tilde{p}}(t),\hat{\tilde{u}}(t)\right) \le \frac{1}{2}\left\|\hat{\tilde{u}}(t)\right\|^2 + 2\left\|\tilde{f}\right\|^2, \tag{4.2}$$

where

$$F_1 = 2\left(d_c(u(t),\tilde{u}(t) + d_c(\tilde{u}(t),u(t)),\hat{\tilde{u}}(t)\right),$$
$$F_2 = 2\left(d_c(\tilde{u}(t),\tilde{u}(t)),\hat{\tilde{u}}(t)\right).$$

Taking inner product of the second equation of (4.1) with $2\hat{\tilde{p}}(t)$

$$\beta\left(\|\tilde{p}(t)\|\right)_{\hat{t}} + 2\left(\nabla\cdot\hat{\tilde{u}}(t),\hat{\tilde{p}}(t)\right) \le \frac{\beta}{2}\left\|\hat{\tilde{p}}\right\|^2 + \frac{2}{\beta}\|\tilde{g}(t)\| \tag{4.3}$$

Combing (4.2) and (4.3), Integration by parts and using Lemma1, Lemma 2, we have

$$\left(\|\tilde{u}(t)\| + + \beta\|\tilde{p}(t)\|\right)_{\hat{t}} + 2\nu\left|\hat{\tilde{u}}(t)\right|_1^2 + \sum_{j=1}^{2}F_j \le \frac{1}{2}\left\|\hat{\tilde{u}}(t)\right\|^2 + \frac{\beta}{2}\left\|\hat{\tilde{p}}(t)\right\|^2 + 2\left\|\tilde{f}\right\|^2 + \frac{2}{\beta}\|\tilde{g}(t)\|^2 \tag{4.4}$$

Now we are going to estimate $|F_j|$

$$|F_1| \le \frac{\nu}{2}\left|\hat{\tilde{u}}\right|_1^2 + \frac{c}{2}\|u\|_\infty^2\|\tilde{u}\|^2,$$
$$|F_2| \le \frac{\nu}{2}\left|\hat{\tilde{u}}\right|_1^2 + \frac{cM^2N}{2}\|\tilde{u}\|^4.$$

By substituting the above estimation in (4.5), we gat

$$\left(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}\|^2\right)_{\hat{t}} + \nu\left|\hat{\tilde{u}}\right|_1^2 \le \frac{1}{2}\|\tilde{u}(t)\|^2 + \frac{\beta}{2}\left\|\hat{\tilde{p}}(t)\right\|^2 + \frac{c}{\nu}\|u\|_\infty^2\|\tilde{u}(t)\|^2 + \frac{cM^2N}{\nu}\|\tilde{u}(t) + 2\left\|\tilde{f}(t)\right\|^2 + \frac{2}{\beta}\|\tilde{g}(t)\|^2, \tag{4.6}$$

let

$$M_1 = c + \frac{c}{\nu}\|u\|_\infty^2, \qquad M_2 = \frac{cM^2N}{\nu},$$

$$E(t) = \|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2 + 4\nu\tau\sum_{t'=\tau}^{t-\tau}\left|\hat{\tilde{u}}(t')\right|_1^2$$

$$\rho(t) = 3\left(3\|\tilde{u}(0)\|^2 + \beta\|\tilde{p}(0)\|^2\right) + 2\left(3\|\tilde{u}(\tau)\|^2 + \beta\|\tilde{p}(\tau)\|^2\right) + 8\tau\sum_{t'=\tau}^{t-\tau}\left(3\|\tilde{f}(t')\|^2 + \frac{1}{\beta}\|\tilde{g}(t')\|^2\right)$$

Then by letting $\tau \le 1$ and summing (4.6) for $t \in R_\tau$. We have from lemma (2.1) that

$$E(t) \le \rho(t) + \tau\sum_{t'=\tau}^{t-\tau}\left[M_1 E(t') + M_2 E^2(t')\right],$$

We use the lemma (2.4) to obtain the following results

**Theorem 1:** There exit positive constant $M_1$ and $M_3$ depending only on $\|u\|_\infty$ and $\nu$, such that, if for some $t_1 \in R_\tau$.

$$\rho(t_1)e^{2M_1 t_1} \le \frac{M_3}{M^2 N},$$

then for all $t \in R_\tau$, $t \le t_1$ $E(t) \le \rho(t)e^{2M_1 t}$.

**Convergence:** Let U, P be the solution of (1.1) and u, p be the solution of (3.1). Define $U^N = P_N U$, $P^N = P_N U$, $e = U^N - U$, $w = P^N - P$. we derive from (1.1) and (3.1) that

$$\begin{cases} e_{\hat{t}}(t) + d(e(t), U^N + e(t)) + d(U^N, e) - \nu\nabla^2\hat{e}(t) + \nabla\hat{w}(t) \\ = E_1 + E_2 + \nabla E_3 + E_4, \beta w_{\hat{t}}(t) + \nabla\cdot\hat{e} = \beta E_5 + \nabla\cdot E_6, \\ e(0) = (P_c - P_N)U_0, \quad e(\tau) = (P_c - P_N)(U_0 + \tau\partial_t U(0) - U(\tau)), \\ w(0) = (P_c - P_N)P_0, \quad w(\tau) = (P_c - P_N)(P_0 + \tau\partial_t P(0) - P(\tau)). \end{cases} \quad (5.1)$$

where

$$E_1(t) = \frac{\partial U^N}{\partial t} - U_{\hat{t}}^N,$$

$$E_2(t) = (U^N\cdot\nabla)U^N - d(U^N, U^N),$$

$$E_3(t) = P^N - \hat{P}^N,$$

$$E_4 = (P_c - P_N)f,$$

$$E_6(t) = P_{\hat{t}}^N,$$

$$E_6(t) = U^N - \hat{U}^N,$$

we have to estimate the right hand term in (5.1)

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_1(t')\|^2 \le c\tau^4\|P^N\|^2_{H^1(0,T;L^2(\Omega))},$$

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_2(t')\|^2 \le c\tau^3\left\|\frac{\partial U}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega))},$$

$$\|U\|^2_{\mu+1} + cN^{-2\mu}\|U\|^4_{\mu+1}$$

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_3(t')\|^2 \le c\tau^4\|P^N\|^2_{H^2(0,T;L^2(\Omega))},$$

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_4(t')\|^2 \le cN^{-2\mu}\|f^N\|^2_{L^2(0,T;H^\mu(\Omega))},$$

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_5(t')\|^2 \le c\beta\tau^{-1}\|P^N\|^2_{H^1(0,T;L^2(\Omega))},$$

$$\tau\sum_{t'=\tau}^{t-\tau}\|E_6(t')\|^2 \le c\tau^4\|U^N\|^2_{H^1(0,T;H^1(\Omega))},$$

$$\|e(\tau)\|^2 \le c\tau^4\|U^N\|^2_{H^1(0,T;L^2(\Omega))},$$

$$\|w(\tau)\|^2 \le c\tau^4\|P^N\|^2_{H^2(0,T;L^2(\Omega))},$$

$$\|e(0)\|_1^2 \le cN^{-2\mu}\|U_0\|^2_{\mu+1},$$

$$\|w(0)\|_1^2 \le cN^{-2\mu}\|P_0\|^2_{\mu+1}.$$

Suppose that,

$\tau = O(N^{-S})$, $\quad c_3\tau^2 \le \beta \le c_4\tau^2$

where $c_3$ and $c_4$ are positive constant. If s>2 then

$$\beta^{-1}\left(\tau^4 + N^{-2s}\right) + \beta \le \frac{c}{M^2 N}.$$

By an argument as in Theorem 1, we get the following result

**Theorem 2:** Assume $\tau = O(N^{-S})$, with $s > 2$, $\mu > n/2$ and that

$$U \in H^3(0,T;L^2(\Omega)) \cap H^2(0,T;H^1(\Omega)) \cap C(0,T;H^{\mu+1}), P \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap C(0,T;H^{\mu+1}), f \in L^2(0,T;H^\mu)$$

,Then for all $t \le T$

$$\|U(t) - u(t)\|^2 \le M_4(\beta^{-1}(\tau^4 + N^{-2s}),$$

where $M_4$ is positive constant depending only on $\nu$ and the norm of **U** and **P** in the space mentioned in he above.

**The Numerical Results:** In this section, we examine the numerical performances. We choose the function f in such way that the solution of (1.1) is of the form

$$\begin{cases} U_1 = Cosx_2\exp(Sinx_1 + Sinx_2 + 0.1t) \\ U_2 = -Cosx_2\exp(Sinx_1 + Sinx_2 + 0.1t) \\ P = -(Cos2x_1 + Cos2x_2)\exp(0.2t) \end{cases}$$

The error for the speed and pressure are defined by

$$E(U_i(t)) = \left(h^2\frac{\sum_{x\in\Omega_N}|U_i(t) - u_i(t)|^2}{\sum_{x\in\Omega_N}|U_i(t)|^2}\right)^{1/2} \quad i = 1,2$$

$$E(P(t)) = \left(h^2\frac{\sum_{x\in\Omega_N}|P(t) - p(t)|^2}{\sum_{x\in\Omega_N}|P(t)|^2}\right)^{1/2}$$

where $u_i$ and p are the solution of (3.1). We solve (1.1) by the scheme (3.1). The numerical results are tabulated in Table 1,2,3 and 4. The results are very accurate, if $\beta$ is chosen suitably, Table 1 and 2. Table 3 and 4 show that $\beta$ become less, the accuracy decreases. It agree with our theoretical analysis very well, since the nonlinear term are computed on the collocation points, this method is performed very simple.

Table 1: The Error of (3.1), $\tau = 0.01$, $\nu = 0.001$, $\beta = 0.05$

| t | $E(U_1(t))$ | $E(U_2(t)$ | $E(P(t))$ |
|---|---|---|---|
| 1.000 | 0.27778E-05 | 0.26672E-05 | 0.12323E-05 |
| 2.000 | 0.99599E-05 | 0.99270E-05 | 0.70355E-05 |
| 3.000 | 0.32944E-04 | 0.34099E-04 | 0.25392E-04 |
| 4.000 | 0.12739E-03 | 0.14027E-03 | 0.10581E-03 |
| 5.000 | 0.71757E-03 | 0.78278E-03 | 0.53868E-03 |

Table 2: The Error of (3.1), $\tau = 0.005$, $\nu = 0.0001$, $\beta = 0.05$

| t | $E(U_1(t))$ | $E(U_2(t)$ | $E(P(t))$ |
|---|---|---|---|
| 1.000 | 0.47123E-05 | 0.46291E-05 | 0.15262E-05 |
| 2.000 | 0.17054E-04 | 0.17011E-04 | 0.10597E-04 |
| 3.000 | 0.67010E-04 | 0.68518E-04 | 0.48212E-04 |
| 4.000 | 0.30433E-03 | 0.32366E-03 | 0.22345E-03 |
| 5.000 | 0.19929E-02 | 0.21263E-02 | 0.13431E-02 |

Table3: The Error of (3.1), $\tau = 0.005$, $\nu = 0.001$, $\beta = 0.01$

| t | $E(U_1(t))$ | $E(U_2(t)$ | $E(P(t))$ |
|---|---|---|---|
| 1.000 | 0.48783E-03 | 0.48790E-03 | 0.12465E-03 |
| 2.000 | 0.11966E-02 | 0.11968E-02 | 0.76616E-03 |
| 3.000 | 0.27505E-02 | 0.27504E-02 | 0.22933E-02 |
| 4.000 | 0.83372E-02 | 0.83360E-02 | 0.92043E-02 |
| 5.000 | 0.51157E-01 | 0.51165E-01 | 0.47461E-01 |

Table 4: The Error of (3.1), $\tau = 0.005$, $\nu = 0.001$, $\beta = 0.005$

| t | $E(U_1(t))$ | $E(U_2(t)$ | $E(P(t))$ |
|---|---|---|---|
| 1.000 | 0.12014E-02 | 0.12013E-02 | 0.37021E-03 |
| 2.000 | 0.30977E-02 | 0.30977E-02 | 0.20000E-02 |
| 3.000 | 0.80261E-02 | 0.80258E-02 | 0.11171E-01 |
| 4.000 | 0.10825E+00 | 0.10826E+00 | 0.10842E+00 |
| 5.000 | 0.20355E+02 | 0.20437E+02 | 0.29057E+02 |

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