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Resolvent of Fourth Order Differential Equation in Half Axis

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Abstract: Let H be a seperable Hilbert space and $H_1 = L_2(0,\infty;H)$. The all functions are defined in range $[0,\infty)$, their values belongs to space H, they are measurable in the meaning of Bochner and provides the condition of

$$\int_0^\infty \|f(x)\|^2 dx < \infty \cdot$$

If the scaler product is defined in H_1 by the formula $(f,g)_{H_1} = \int_0^\infty (f,g)_H dx \qquad f(x),\ g(x) \in H_1,$

H_1 forms a seperable Hilbert space. In this study, in space H_1, it is investigated that Green's function (resolvent) of the operator formed by the diferential expression
$(-1)^n y^{(2n)} + Q(x)y, \qquad 0 \le x < \infty,$
and boundary conditions
$y^{(j)}(0) - h_j y^{(j-1)}(0) = 0, \qquad j = 1,3,\dots\dots,2n-1$
where Q(x) is a normal operator that has pure discrete spectrum for every $x \in [0,\infty)$ in H. Assumed that domain of Q(x) is independent from x and resolvent set of Q(x) belongs to $|\arg\lambda - \pi| < \varepsilon \quad (0 < \varepsilon < \pi)$ of complex plane λ. In addition assumed that the operator function Q(x) satisfies the Titchmars-Levitan conditions. h_js are arbitrary complex numbers. The obtained result has been applied to an example.

Key Words: Spectral analysis, Eigenvalues, Eigenfunctions, Operator

Introduction

Let H be a seperable Hilbert space. Assumed that $H_1 = L_2(0,\infty;H)$. The functions are defined in range $[0,\infty)$ and their values belongs to space H, they are measurable in the meaning of Bochner (Yosida, 1980) and provides the condition of

$$\int_0^\infty \|f(x)\|^2 dx < \infty \cdot$$

If the scalar product is defined in H_1 by the equation

$$(f,g)_1 = \int_0^\infty (f(x),g(x))dx \qquad f(x), g(x) \in H_1,$$

H_1 forms a seperable Hilbert space (Balakrishnan, 1976). Where $\|.\|$, $(.,.)$ are norm and scalar product in H, respectively.

In this work, in space $H_1 = L_2(0,\infty;H)$, it is investigated that Green's fuction(resolvent) of operator L formed by the differential expression

$$y^{(iv)} + Q(x)y \quad , \qquad 0 \le x < \infty \tag{1}$$

and the boundary conditions

$$y'(0) - h_1 y(0) = 0, \tag{2}$$

$$y'''(0) - h_2 y''(0) = 0$$

where Q(x) is a normal operator for every $x \in [0,\infty)$ in H and its inverse is a compact operator, h_1, h_2 are arbitrary complex numbers.

In the case of Q(x) is a normal operator and the boundary conditions are $y^{(j)}(0) = 0 \quad (j=1,2,\dots,n)$, resolvent of operator L was investigated by Aliyev and Bayramoglu (1981). Green's fuction of Sturm-Liouville equation with infinity operator coefficient was first investigated by Levitan (1968).

In the space $L_2(-\infty,\infty;H)$, Green's function and the asymptotic behaviour for the number of the eigenvalues of the operator formed by differential expression

$$(-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)}$$

was obtained by Bayramoğlu (1971), where $Q_j(x)$ (j=2,...,2n) are the self-adjoint operators in H. Later on many studies (Aslanov, 1994; Boymatov, 1973; Kleiman, 1977; Otelbayev, 1990 and Saito, 1975) were published in this subject. Wide reference of these studies is given in (Kostyuchenko and Sargsyan, 1979 and Otelbayev, 1977). The main reference related to Green's function for the ordinary differential equation is Stakgold's book. (Stakgold, 1998).

Determination of the Problem: Resolvent of the operator formed by differential expression (1) and boundary conditions (2) in $H_1 = L_2(0,\infty;H)$ will be investigated.

$$y^{iv} + Q(x)y + \mu y \tag{3}$$

$$y'(0) - h_1 y(0) = 0 \tag{4}$$

$$y'''(0) - h_2 y''(0) = 0 \tag{5}$$

where $\mu > 0$ is a real number. It is assumed that Q(x) is a normal operator mapping in H for every $x \in [0,\infty)$ and satisfies the following specification:

- Let Q(x) be a normal operator for each $x \in [0,\infty)$ in H, and $\overline{D} = H$, where D(Q(x))=D independent from x. (Here $\overline{D}$ shows the closure of D)

- Let $Q^{-1}(x)$ be a compact operator for every x $(Q^{-1}(x) \in \sigma_\infty)$ and $1 \le |\alpha_1(x)| \le |\alpha_2(x)| \le \ldots\ldots \le |\alpha_n(x)| \le \ldots\ldots$ where $\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots\ldots,$ are the eigenvalues of Q(x).
- Assumed that resolvent set of Q(x) belongs to $s_\varepsilon = \{\lambda : \pi - \varepsilon < \arg\lambda < -\pi + \varepsilon\ ,\ 0 < \varepsilon < \pi\}$ of complex plane λ.
- Suppose that function $F(x) = \sum_{k=1}^{\infty} \frac{1}{|\alpha_k(x)|^{7/4}}$ belongs to

$$L(0,\infty): \int_0^\infty F(x)dx < \infty$$

- $\|Q^{-1/4}(x).Q^{1/4}(s)\| \le c$ and $\|Q^{1/4}(x).Q^{-1/4}(s)\| \le c$ while $|x - s| \le 1$, c=constant
- Assumed that $\|Q^{-a}(x)[Q(s) - Q(x)]\| \le c|x - s|$ while $|x - s| \le 1$, where c=constant and $0 < a < \frac{5}{4}$. (c denotes different constants)

In (Kostyuchenko and Levıtan, 1967) some examples are shown that these conditions satisfied. Let B(H) be a Banach space whose elements are bounded operators mapping in H (Kato, 1980) $G(x,s;\mu)$ function which belongs to B(H) for $0 \le x, s < \infty$ and satisfies the following conditions is called Green's function of problem (3)-(5).

- The operator function $G(x,s;\mu)$ itself and its two partial derivative are continuous functions for variables x and s $(0 \le x, s \le \infty)$.
- When $s \ne x$ third derivative of $G(x,s;\mu)$ for s is continuous.
- $\frac{\partial^3 G}{\partial s^3}(x, x+0, \mu) - \frac{\partial^3 G}{\partial s^3}(x, x-0, \mu) = I$ (I is identity operator in H)
- When $s \ne x$,

$$\frac{\partial^4 G}{\partial s^4}(x,s;\mu) + G(x,s;\mu)Q(s) + \mu G(x,s;\mu) = 0$$

$$\left.\frac{\partial G}{\partial s}(x,s;\mu)\right|_{s=0} - h_1 G(x,s;\mu)\big|_{s=0} = 0,$$

$$\left.\frac{\partial^3 G}{\partial s^3}(x,s;\mu)\right|_{s=0} - h_2 \left.\frac{\partial^2 G}{\partial s^2}(x,s;\mu)\right|_{s=0} = 0$$

According to parametrics method, the operator function $G(x;s;\mu)$ will be found as a solution of integral equation given by

$$G(x,s;\mu) = r(x-s)g(x,s;\mu) - \int_0^\infty \{r^{(IV)}(x-\xi)g(x,\xi;\mu) + 4r'''(x-\xi)g'(x,\xi;\mu) + 6r''(x-\xi)g''(x,\xi;\mu) + 4r'(x-\xi)g'''(x,\xi;\mu) + r(x-\xi)g(x,\xi;\mu)[Q(\xi) - Q(x)]\}G(\xi,s;\mu)d\xi \tag{6}$$

where

$$r(u) = \begin{cases} 1 & |u| \le \rho \\ 0 & |u| \ge 2\rho, \quad 0 < \rho < \frac{1}{2} \end{cases}$$

is any fixed sufficiently smooth function and

$$g(x,s;\alpha) = \frac{\sqrt{2}}{8}\alpha^{-3}(1+i)e^{-\alpha\frac{\sqrt{2}}{2}[|x-s|+i|x-s|]} - \frac{\sqrt{2}}{8}\alpha^{-3}(-1+i)e^{-\alpha\frac{\sqrt{2}}{2}[|x-s|-i|x-s|]}$$
$$+ \frac{\left[\frac{1}{4}(1+i)h_2 + \frac{\sqrt{2}}{4}\alpha^{-1}(-1+i)h_1h_2 - \frac{\sqrt{2}}{4}\alpha(-1+i) + \frac{1}{4}(1+i)h_1\right]}{\alpha^2 i\left(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2\right)} \cdot e^{-\alpha\frac{\sqrt{2}}{2}(1+i)(x+s)}$$
$$+ \frac{\left[-\frac{1}{2}h_2 + \frac{\sqrt{2}}{4}\alpha(-1+i) + \frac{1}{2}h_1\right]}{\alpha^2 i\left(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2\right)} e^{\alpha\frac{\sqrt{2}}{2}[-(x+s)+i(-x+s)]}$$
$$+ \frac{\left[-\frac{1}{2}h_1 - \frac{\sqrt{2}}{4}\alpha i + \frac{1}{2}h_2\right]}{\alpha^2 i\left(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2\right)} e^{\alpha\frac{\sqrt{2}}{2}[-(x+s)+i(x-s)]}$$
$$+ \frac{\left[\frac{1}{4}(-1+i)h_1 + \frac{\sqrt{2}}{4}\alpha^{-1}(1+i)h_1h_2 - \frac{\sqrt{2}}{4}\alpha(1+i) + \frac{1}{4}(-1+i)h_2\right]}{\alpha^2 i\left(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2\right)} \cdot e^{\alpha\frac{\sqrt{2}}{2}(-1+i)(x+s)},$$

$$0 \le x\ ,\ s < \infty. \tag{7}$$

Let's say that $g = g_1 + g_2 + g_3 + g_4 + g_5 + g_6$. $\alpha = \sqrt[4]{Q(x) + \mu I}$ and defined by formula of the spectral expansion

$$\sqrt[4]{Q(x) + \mu I} = \sum_{n=1}^{\infty} \sqrt[4]{\alpha_n(x) + \mu}\,(., e_n)e_n.$$

where $e_1(x), e_2(x), \ldots, e_n(x), \ldots$ are respectivly the ortonormalized eigenvectors corresponding to eigenvalues $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \ldots$ of Q(x). $\sqrt[4]{\alpha_j(x) + \mu}$'s are defined from $-\pi + \varepsilon < \arg(\alpha_n(x) + \mu) < \pi - \varepsilon$. Note that,

$$|\lambda + \mu| \ge |\mu|\mathrm{Sin}\varepsilon, \qquad |\lambda + \mu| \ge |\lambda|\mathrm{Sin}\varepsilon$$

is satisfied for $\lambda \in S_\varepsilon$, $\mu > 0$.

It will be shown that integral equation (6) has only one solutions and this solution is Green's fuction of the problem (3)-(5).

Equation (6) will be investigated in these spaces; X_2, $X_3^{(1)}$, $X_4^{(-1/4)}$, X_5. These are shown that they are Banach spaces and given by Levitan (1968).

Consider that integral Eq. (6) is in the space X_2. Let us show that this equation has one solution in X_2 for $\mu >> 0$ (μ is a big enough positive value) and the solution can be found by successive aproximation method. For this it is enough to show that $g(x,s;\mu)\in X_2$ and the operator of

$$NA=\int_0^x\{r^{(IV)}(x-\xi)g(x,\xi;\mu)+4r'''(x-\xi)g'(x,\xi;\mu)+6r''(x-\xi)g''(x,\xi;\mu)$$

$$+4r'(x-\xi)g'''(x,\xi;\mu)+r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\}A(\xi,s;\mu)d\xi$$

is constriction operator in X_2 for $\mu >> 0$.

Lemma 1: If operator function Q(x) satisfies the conditions 4-) and 6-) for $\mu >> 0$, operator N is constriction operator in the space X_2.

Proof: If it is shown that the norm of operator N for $\mu >> 0$ are small enough, it is demonstrated that N is constriction operator for $\mu >> 0$.

$$NA=\sum_{j=1}^{5}N_jA$$

$$N_1A=\int_0^\infty[r(x-\xi)g(x,\xi;\mu)[Q(x)-Q(\xi)]]A(\xi,\eta)d\xi$$

$$N_2A=\int_0^\infty 4r'(x-\xi)g'''(x,\xi;\mu)A(\xi,\eta)d\xi$$

$$N_3A=\int_0^\infty 6r''(x-\xi)g''(x,\xi;\mu)A(\xi,\eta)d\xi$$

$$N_4A=\int_0^\infty 4r'''(x-\xi)g'(x,\xi;\mu)A(\xi,\eta)d\xi$$

$$N_5A=\int_0^\infty r^{(IV)}(x-\xi)g(x,\xi;\mu)A(\xi,\eta)d\xi$$

$\|N\|\le\|N_1\|+\|N_2\|+\|N_3\|+\|N_4\|+\|N_5\|$ can be written from nature of norm.

Let's do the operations for operator N_1A.

$$g(x,s;\mu)=g_1+g_2+g_3+g_4+g_5+g_6$$

$$N_1A=\int_0^\infty[r(x-\xi)g(x,\xi;\mu)[Q(x)-Q(\xi)]]A(\xi,\eta)d\xi$$

$$N_1A=\int_0^\infty\left[r(x-\xi)\sum_{i=1}^{6}g_i(x,\xi;\mu)[Q(x)-Q(\xi)]\right]A(\xi,\eta)d\xi$$

$$N_1A=N_{11}A+N_{12}A+N_{13}A+N_{14}A+N_{15}A+N_{16}A$$

$$\|N_1\|\le\|N_{11}\|+\|N_{12}\|+\|N_{13}\|+\|N_{14}\|+\|N_{15}\|+\|N_{16}\|$$

$$N_1A=\int_0^x r(x-\xi)g(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)d\xi$$

$$N_{11}A=\int_0^\infty r(x-\xi)g_1(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)d\xi$$

$$=\int_{|x-\xi|\le1}r(x-\xi)g_1(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)d\xi$$

$$+\int_{|x-\xi|>1}r(x-\xi)g_1(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)d\xi=b_1+b_2$$

$$\|N_{11}A(x,\xi)\|_2=\|b_1+b_2\|_2\le\|b_1\|_2+\|b_2\|_2$$

$$\|N_{11}A(x,\xi)\|_2^2\le2\left(\|b_1\|_2^2+\|b_2\|_2^2\right).$$

where $b_2=0$ according to function $r(x-\xi)$.

$$\|b_1\|_2^2=\left\|\int_{|x-\xi|\le1}r(x-\xi)g_1(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)d\xi\right\|_2^2$$

$$\le\left[\int_{|x-\xi|\le1}\|r(x-\xi)g_1(x,\xi;\mu)[Q(x)-Q(\xi)]A(\xi,\eta)\|_2\,d\xi\right]^2 \quad (9)$$

$$\|b_1\|_2^2\le c^2\mu^{2q}\left[\int_{|x-\xi|\le1}|x-\xi|^{-\varepsilon}\|A(\xi,\eta)\|_2\,d\xi\right]^2$$

is found. Hence

$$\|b_1\|_{X_2}^2\le\int_0^x\int_0^x c\mu^{2q}\left[\int_{|x-\xi|\le1}|x-\xi|^{-\varepsilon}\|A(\xi,\eta)\|_2\,d\xi\right]^2dxd\eta$$

is obtained, or

$$\|b_1\|_{X_2}\le c\mu^{2q}\|A(x,\eta)\|_{X_2}<\infty.$$

Therefore,

$$\|N_{11}A\|_2^2\le c\mu^{2q}\|A(x,\eta)\|_{X_2}$$

is derived. Here, q<0 is constant. Thus, operator $N_{11}A(x,\xi)$ is bounded with small enough norm of large $\mu>0$. In a similar way, operators $N_{12}A, N_{13}A, N_{14}A, N_{15}A, N_{16}A$ are bounded with small enough norm of large $\mu>0$. Then it is obtained that operator NA is constriction operator in space X_2 for $\mu >> 0$. Thus Lemma 1 is proved.

If it can be shown that $r(x-s)g(x,s;\mu)$ belongs to the space X_2, then, it is obtained that Eq. (6) has only one solution belonging to X_2 for $\mu >> 0$.

$$\|g\|_2 \le \|g_1\|_2 + \|g_2\|_2 + \|g_3\|_2 + \|g_4\|_2 + \|g_5\|_2 + \|g_6\|_2$$

can be written.

Now let us show that **rg** belongs to space X_2 assuming that the condition 4-) of function Q(x) is fulfilled. Let us perform the operation for any term included by **rg,** for example the term $r(x-s)g_1(x,s;\mu)$. That is, let us show that $rg_1 \in X_2$. In a same manner, it is indicated that other terms also belong to X_2. Since $r(x-s)g_1(x,s;\mu)$ is a function of normal operator valued function Q(x), using the spectral expansion formula for normal operators(Yosida, 1980):

$$\|rg_1\|_2^2 = \sum_{j=1}^{\infty}\left(\frac{\sqrt{2}}{8}\right)^2 \left| r(x-s)(1+i)\left(\alpha_j(x)+\mu\right)^{-3/4} e^{-\left(\alpha_j(x)+\mu\right)^{1/4}\frac{\sqrt{2}}{2}(1+i)|x-s|}\right|^2$$

$$\|rg_1\|_2^2 \le (1/16)\sum_{j=1}^{\infty} \cdot \left|\alpha_j(x)+\mu\right|^{-3/2} e^{-\sqrt{2}\delta|x-s|\operatorname{Re}\left(\alpha_j(x)+\mu\right)^{1/4}}$$

$$\delta = \cos t > 0$$

is implied. From the 4-). property of Q(x)

$$\int_0^{\infty}\int_0^{\infty} \|rg_1\|_2^2 \, ds\,dx = c\int_0^{\infty}\sum_{j=1}^{\infty} \frac{dx}{\left|\alpha_j(x)+\mu\right|^{7/4}} < \infty,$$

$$c = \text{const} > 0$$

is obtained. Thus it is denoted that $rg_1 \in X_2$. Therefore the following theorem has been proved.

Theorem 1: If the conditions 4-) and 6-) of operator Q(x) are satisfied, then, for $\mu >> 0$, there exists a solution in the space X_2 for Eq. (6) and it is unique. This solution can be found by successive aproximation method.

The following lemma can be proved.

Lemma 2: If operator fuction **Q(x)** satisfies the conditions in Lemma 1 then for $\mu >> 0$, operator **N** is a constriction operator in every spaces $X_2, X_3^{(1)}, X_4^{(-1/4)}$ and X_5. In addition to the conditions 1-) and 6-), if operator fuction Q(x) satisfies the condition $\left\|Q^{1/4}(x)Q^{-1/4}(s)\right\| \le c$, c=constant, then $g \in X_4^{(-1/4)}$.

Derivations of Green's Function: Let us try to show that operator function $G(x,s;\mu)$ has the derivatives $\frac{\partial^j G(x,s;\mu)}{\partial s^j}$ (j=1,2,3). If the derivatives of both sides of Eq. (6) is calculated according to s

$$\frac{\partial^j G(x,s;\mu)}{\partial s^j} = \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j} - \int \{ r^{(iv)}(x-\xi)g(x,\xi;\mu) + 4r'''(x-\xi)g'(x,\xi;\mu)$$

$$+ 6r''(x-\xi)g''(x,\xi;\mu) + 4r'(x-\xi)g'''(x,\xi;\mu)$$

$$+ r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\} \frac{\partial^j G(\xi,s;\mu)}{\partial s^j} d\xi \quad (10)$$

$$K_j(x,s;\mu) = \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j} - NK_j(\xi,s;\mu) \quad \cdot \quad (j=1,2,3) \quad (11)$$

can be written. Let us investigate integral Eq. (11) in Banach space $X_3^{(1)}$. In Lemma 2, N was denoted is a constriction operator in the space $X_3^{(1)}$ for $\mu >> 0$. If it is implied that operator function $\frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$, (j=1,2,3) belongs to $X_3^{(1)}$, it is shown that there exists a solution for Eq. (11) in $X_3^{(1)}$ for $\mu >> 0$. It is seen that $\frac{\partial^j rg}{\partial s^j} \in X_3^{(1)}$, that is,

$$\sup_{0 \le x < \infty} \int_0^{\infty} \left\| \frac{\partial^j (rg)}{\partial s^j} \right\|_H^p ds < \infty$$

from clear expression of operator function $g(x,s;\mu)$. It is demonstrated that $\frac{\partial^j G(x,s;\mu)}{\partial s^j} - \frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$ (j=1,2,3) is a continious function for $\mathbf{s}(s \ne x)$, performing the similar operations as in (Levıtan, 1968) and (Bayramoglu, 1971). On the other hand since $\frac{\partial^j [r(x-s)g(x,s;\mu)]}{\partial s^j}$ (j=1,2) is continuous, function $\frac{\partial^j G(x,s;\mu)}{\partial s^j}$ (j=1,2) is also continuous for according to s.

From $\frac{\partial^3 [r(x-s)g(x,s;\mu)]}{\partial s^3}$, it is denoted that this function satisfies the condition $\frac{\partial^3 [rg(x,x+0;\mu)]}{\partial s^3} - \frac{\partial^3 [rg(x,x-0;\mu)]}{\partial s^3} = I$ at the point s=x. This results in that operator function $\frac{\partial^3 G}{\partial s^3}$ fulfilles the condition 4-) from the continuity of $\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 rg}{\partial s^3}$. The following Lemma can be proved.

Lemma 3: Assume that operator function Q(x) satisfies the conditions 1-), 3-) and

$$\left\|Q^{1/4}(x)Q^{-1/4}(s)\right\| \le c \quad (12)$$

while $|x-s| \le 1$. In this case;

$$\frac{\partial^4[r(x-s)g(x,s;\mu)]}{\partial s^4} \in X_4^{-1/4}$$

that is,

$$\sup_{0\le x<\infty}\int_0^\infty \left\| \frac{\partial^4[r(x-s)g(x,s;\mu)]}{\partial s^4} Q^{-1/4}(s)\right\| ds < \infty .$$

The Fourth Derivative of Green's Function: In previous part it has been shown that the derivative $\frac{\partial^3 G}{\partial s^3}$ of Green's function $G(x,s;\mu)$ belongs to the space X_3 and it satisfies the continuity $(x \neq s)$ for the variable **s** and the following expression

$$\frac{\partial^3 G(x,s;\mu)}{\partial s^3} = \frac{\partial^3[r(x-s)g(x,s;\mu)]}{\partial s^3} - \int_0^\infty P(x,\xi;\mu)\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} d\xi \quad (13)$$

where

$$P(x,\xi;\mu) = r^{(iv)}(x-\xi)g(x,\xi;\mu) + 4r'''(x-\xi)g'(x,\xi;\mu) + 6r''(x-\xi)g''(x,\xi;\mu)$$

$$+4r'(x-\xi)g'''(x,\xi;\mu) + r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)].$$

Let us write Eq. (10) as follows

$$L(x,s;\mu) = l(x,s;\mu) - \int_0^\infty P(x,\xi;\mu)L(\xi,s,\mu)d\xi . \quad (14)$$

Here

$$L(x,s;\mu) = \frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 rg}{\partial s^3}$$

and

$$l(x,s;\mu) = -\int_0^\infty P(x,\xi;\mu)\frac{\partial^3 (rg)(\xi,s;\mu)}{\partial s^3} d\xi .$$

Let us derive the Eq. (14) according to **s** as formal. From this

$$\frac{\partial L(x,s;\mu)}{\partial s} = \frac{\partial l(x,s;\mu)}{\partial s} - \int_0^\infty P(x,\xi;\mu)\frac{\partial L(\xi,s;\mu)}{\partial s} d\xi$$

is obtained. If the expression

$$\frac{\partial^3[r(x-(x+0))g(x,x+0;\mu)]}{\partial s^3} - \frac{\partial^3[r(x-(x-0))g(x,x-0;\mu)]}{\partial s^3} = I$$

is used and if we write as

$$\frac{d}{d\eta}\left(\int_0^{s-0} P(x,\xi;\mu)\frac{\partial^3(rg)(\xi,s;\mu)}{\partial s^3}d\xi - \int_{s+0}^\infty P(x,\xi;\mu)\frac{\partial^3(rg)(\xi,s;\mu)}{\partial s^3}d\xi\right),$$

$$\frac{\partial l(x,s;\mu)}{\partial s} = -P(x,s;\mu) - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4(rg)(\xi,s;\mu)}{\partial s^4} d\xi$$

is found. Let us say that

$$\frac{\partial l(x,s;\mu)}{\partial s} = l_1(x,s;\mu) .$$

If it can be shown that element l_1 belongs to $X_4^{(-1/4)}$, according to Lemma 2, it is obtained that there exists a derivative of the function $\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 (rg)}{\partial s^3}$ according to **s** and $\frac{\partial}{\partial s}\left(\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 (rg)}{\partial s^3}\right) \in X_4^{(-1/4)}$. From this point, according to Lemma 3, $\frac{\partial^4 G}{\partial s^4} \in X_4^{(-1/4)}$ is obtained. It is found that the element l_1 belongs to $X_4^{(-1/4)}$ by the studies (Levıtan, 1968), (Bayramoglu, 1971).

Satisfying Differential Equation of Green's Function: Let us show that Green's function $G(x,s;\mu)$ for $x \neq s$ satisfies the equation

$$\frac{\partial^4 G}{\partial s^4} + G(x,s;\mu)[Q(s)+\mu I] = 0 .$$

Let $f \in D$. Then,

$$\frac{\partial^4 G}{\partial s^4}(f) + rg[Q(x)+\mu I](f) = -rg[Q(s)-Q(x)](f) - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}(f)d\xi$$

or

$$\frac{\partial^4 G}{\partial s^4}(f) = -rg[Q(s)+\mu I](f) - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}(f)d\xi \quad (15)$$

is obtained. Let $[Q(s)+\mu I]f = \alpha$. From this, Eq. (15) becomes as follows.

$$\frac{\partial^4 G}{\partial s^4}[Q(s)+\mu I]^{-1}\alpha = -rg\alpha - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}[Q(s)+\mu I]^{-1}\alpha d\xi$$

Compairing this equation with Eq. (6),

$$\frac{\partial^4 G}{\partial s^4}\left\{-[Q(s)+\mu I]^{-1}\alpha\right\} = G(x,s;\mu)\alpha$$

is found. From the last expression fourth property is obtained as elements' set of α for every constant $s \ge 0$ is dence everywhere in H.

Satisfaction of Boundary Conditions: Let us show that $G(x,s;\mu)$ satisfies the conditions

$$\left.\frac{\partial G(x,s;\mu)}{\partial s}\right|_{s=0} - h_1 G(x,s;\mu)\Big|_{s=0} = 0$$

$$\left.\frac{\partial^3 G(x,s;\mu)}{\partial s^3}\right|_{s=0} - h_2 \left.\frac{\partial^2 G(x,s;\mu)}{\partial s^2}\right|_{s=0} = 0$$

that is Green's function fulfilles the condition 5-).

$$G(x,s;\mu) = r(x-s)g(x,s;\mu) - \int_0^\infty g(x,\xi;\mu)[Q(\xi)-Q(x)]r(x-\xi)G(\xi,s;\mu)d\xi \quad (16)$$

$$\frac{\partial G(x,s;\mu)}{\partial s} = \frac{\partial[r(x-s)g(x,s;\mu)]}{\partial s} - \int_0^\infty g(x,\xi;\mu)[Q(\xi)-Q(x)]r(x-\xi)\frac{\partial G(\xi,s;\mu)}{\partial s}d\xi \quad (17)$$

From the equations (16) and (17);

$$\left.\frac{\partial r(x-s)g(x,s;\mu)}{\partial s}\right|_{s=0} - \left.\int_0^\infty g(x,\xi;\mu)[Q(\xi)-Q(x)]r(x-\xi)\frac{\partial G(\xi,s;\mu)}{\partial s}d\xi\right|_{s=0}$$

$$-h1\left[r(x-s)g(x,s;\mu) - \int_0^\infty g(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]r(x-\xi)G(\xi,s;\mu)d\xi\right]_{s=0} = 0 \quad (18)$$

is obtained. Considering that.

$$\left.\frac{\partial(rg)}{\partial s}\right|_{s=0} - h_1 rg|_{s=0} = 0$$

from Eq. (18);

$$\int_0^x r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\left[\frac{\partial G(\xi,s;\mu)}{\partial s} - h_1 G(\xi,s;\mu)\right]_{s=0} d\xi = 0 \quad (19)$$

can be written. Homogen equation (19) can be written as below.

$$N\left[\frac{\partial G}{\partial s} - h_1 G\right]_{s=0} = 0$$

Since operator N is constriction operator for $\mu >> 0$, then

$$\left.\frac{\partial G(\xi,s;\mu)}{\partial s} - h_1 G(\xi,s;\mu)\right|_{s=0} = 0$$

is obtained. Thus the first boundary conditions of 5-) is satisfied.

Now let us calculate the thirth derivation of $G(x,s;\mu)$ according to **s**.

$$\left.\frac{\partial^3 G}{\partial s^3}\right|_{s=0} = \left.\frac{\partial^3 (rg)}{\partial s^3}\right|_{s=0} - \left.\int_0^\infty r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3}d\xi\right|_{s=0}$$

$$\left.\frac{\partial^3 (rg)}{\partial s^3}\right|_{s=0} - \left.\int_0^\infty r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3}d\xi\right|_{s=0}$$

$$-h_2\left[\left.\frac{\partial^2 (rg)}{\partial s^2}\right|_{s=0} - \left.\int_0^\infty r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\frac{\partial^2 G(\xi,s;\mu)}{\partial s^2}d\xi\right|_{s=0}\right] = 0 \quad (20)$$

From the expression of $g(x,s;\mu)$, considering that

$$\left.\frac{\partial^3 (rg)}{\partial s^3}\right|_{s=0} - h_2 \left.\frac{\partial^2 (rg)}{\partial s^2}\right|_{s=0} = 0$$

from the equation (20)

$$-\int_0^\infty r(x-\xi)g(x,\xi;\mu)[Q(\xi)-Q(x)]\left[\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} - h_2\frac{\partial^2 G(\xi,s;\mu)}{\partial s^2}\right]_{s=0} d\xi = 0$$

is found.This homogen equation can be expressed by

$$N\left.\left[\frac{\partial^3 G}{\partial s^3} - h_2\frac{\partial^2 G}{\partial s^2}\right]\right|_{s=0} = 0.$$

Since N is constriction operator for $\mu >> 0$

$$\left.\frac{\partial^3 G(x,s;\mu)}{\partial s^3} - h_2\frac{\partial^2 G(x,s;\mu)}{\partial s^2}\right|_{s=0} = 0$$

is obtained. Thus the second condition of 5-) is also fulfilled.

Consequently, it is shown that operator function $G(x,s;\mu)$ satisfies all properties of Green's function.

If integral operator

$$Af = \int_0^\infty G(x,s;\mu)f(s)ds \qquad , \qquad \mu > 0$$

is formed in H_1 by using Green's function obtained, it is seen that A is a Hilbert-Schmidt (H-S) type operator from the property proved

$$\int_0^\infty \int_0^\infty \|G(x,s;\mu)\|_2^2 dxds < \infty.$$

If $Q(x) = Q^*(x)$, $h_1 = h_2$ are real numbers then $G^*(x,s;\mu) = G(s,x;\mu)$ can be proved.

Example: Proof of existing the operator valued function Q(x) satisfied 1-) - 6-) conditions.

Let $H = l_2$,
$$Q(x) = i\begin{pmatrix} q_1(x) & 0 & 0 & \dots & 0 & \dots \\ 0 & q_2(x) & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_j(x) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Where $q_j(x) = K_j x + d$, d=sbt>0, any pozitif numbers satisfiying $\sum_{j=1}^{\infty}\frac{1}{K_j} < \infty$ while $K_1 \le K_2 \le \dots \le K_n \le \dots$. Let show that all Q(x)'s conditions in section 2 were provided.

Let pay attention that it is enough to indicate Q(x) satisfies the conditions almost every where.

1. Q(x) is a normal operator for every x>0 in l_2 because of $Q^*(x) = -iQ(x)$.

2. $$Q^{-1}(x) = -i\begin{pmatrix} q_1^{-1}(x) & 0 & 0 & \dots & 0 & \dots \\ 0 & q_2^{-1}(x) & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_n^{-1}(x) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$q_j^{-1}(x) = (K_j x + d)^{-1} \underset{j\to 0}{\to} 0, \quad (x \ne 0).$$

This also demonstrates $Q^{-1}(x)$ is a compact operator in every x, $(x \ne 0)$.

3.

$$(Q(x)+\lambda I)^{-1} = i\begin{pmatrix} (q_1(x)+\lambda)^{-1} & 0 & 0 & \dots & 0 & \dots \\ 0 & (q_2(x)+\lambda)^{-1} & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (q_n(x)+\lambda)^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This statement shows $(Q(x)+\lambda)^{-1}$ is exist and bounded every point in complex plane exterior half axis $(-\infty, 0]$. That is set **s** is a resolvent set of Q(x).4.

$$\int_0^\infty \sum_{j=1}^\infty \frac{1}{q_j^{7/4}(x)} dx = \int_0^\infty \left(\sum_{j=1}^\infty \frac{1}{(K_j x + d)^{7/4}} \right) dx$$

$$= \sum_{j=1}^\infty \frac{1}{K_j} \int_0^\infty \frac{d(K_j x)}{(K_j x + d)^{7/4}}$$

$$= \sum_{j=1}^\infty \frac{1}{K_j} \int_0^\infty \frac{dt}{(t+d)^{7/4}} = \sum_{j=1}^\infty \frac{1}{K_j}\frac{4}{3} = \frac{4}{3}\sum_{j=1}^\infty \frac{1}{K_j} < \infty$$

5. $q_j^{-1/4}(x) q_j^{1/4}(\xi) = (K_j x + d)^{-1/4}(K_j \xi + d)^{1/4}$

$$(K_j x + d)^{-1/4}(K_j \xi + d)^{1/4} \le (K_j x + d)^{-1/4}(K_j x + x + d)^{1/4}$$

$$= \left(1 + \frac{d}{K_j x + d}\right)^{1/4} \le \left(1 + \frac{d}{d}\right)^{1/4} = \sqrt[4]{2}$$

while $|x - \xi| \le 1 \iff -1 \le \xi - x \le 1 \Rightarrow \xi \le x + 1$.

6. $|x - \xi| \le 1$; Let's take a=1 satisfiying condition $0 < a < \frac{5}{4}$. Assumed that $x \ge 1$ without changing generalization.

$$\left|q_j^{-1}(x)\left[q_j(\xi) - q_j(x)\right]\right| = (K_j x + d)^{-1}\left|K_j \xi - K_j x\right|$$

$$= |x - \xi| K_j (K_j x + d)^{-1} = |x - \xi| \left(x + \frac{d}{K_j}\right)^{-1}$$

$$\le |x - \xi| \left(1 + \frac{d}{K_j}\right)^{-1} \le |x - \xi|.$$

Thus $\left|q_j^{-1}(x)\left[q_j(\xi) - q_j(x)\right]\right| \le |x - \xi|$ while $|x - \xi| \le 1$. Therefore it is showed that 1-) - 6-) conditions are satisfied.

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