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Solutions of Initial Value Problems Using Fifth-Order Runge-Kutta Method Using Excel Spreadsheet

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Abstract: Fifth order Runge - Kutta method (Butcher's method) is used in solving initial value problems of the form $\frac{dy}{dx}=f(x,y)$ by the aid of Excel Spread Sheet for different values of step size h, and the true relative percent error ε_t is calculated for every case.

Keywords: Excel, Runge-Kutta, Initial Value Problem and the True Value

Introduction

Microsoft Excel is used throughout the paper, and gives a simple approach in solving initial value problems of the form $\frac{dy}{dx}=f(x,y)$ by using fifth order Runge – Kutta method (Butcher's method). A spreadsheet is made up of cells as follows:

	A	B	C	D	E
1	A1	B1	C1	D1	E1
2	A2	B2	C2	D2	E2
3	A3	B3	C3	D3	E3
4	A4	B4	C4	D4	E4
5	A5	B5	C5	D5	E5

In these boxes we type the data into, the data that can be typed into every cell, (A1, A2,.....,B1,B2,......,E1,E2,....) can be numeric or algebraic.

Fifth-Order Runge-Kutta Method: Butcher's fifth-order Runge-Kutta method gives us very good accurate results, Buchanan *et al* (1992), Mathews *et al* (1999) and Chapra *et al* (1998).

$$y_{i+1}=y_i+\frac{1}{90}(7k_1+32k_3+12k_4+32k_5+7k_6)h$$

Where

$$k_1=f(x_i,y_i)$$

$$k_2=f\left(x_i+\frac{1}{4}h,y_i+\frac{1}{4}k_1h\right)$$

$$k_3=f\left(x_i+\frac{1}{4}h,y_i+\frac{1}{8}k_1h+\frac{1}{8}k_2h\right)$$

$$k_4=f\left(x_i+\frac{1}{2}h,y_i-\frac{1}{2}k_2h+k_3h\right)$$

$$k_5=f\left(x_i+\frac{3}{4}h,y_i+\frac{3}{16}k_1h+\frac{9}{16}k_4h\right)$$

$$k_6=f\left(x_i+h,y_i-\frac{3}{7}k_1h+\frac{2}{7}k_2h+\frac{12}{7}k_3h-\frac{12}{7}k_4h+\frac{8}{7}k_5h\right)$$

In Chapra *et al.* (1998) is stated that higher-order Runge-Kutta formulas such as Butcher's method are available, but in general, beyond fourth-order methods the gain in accuracy is offset by the added computational effort and complexity. But by using Excel this complexity is solved and we will obtain a very good results quickly and easily and also we can change the value of the step size h and we will have the results immediately as we see in the following example.

Example: Use fifth-order Runge-Kutta method by the aid of Excel to solve

$$\frac{dy}{dx}=f(x,y)=4e^{0.8x}-0.5y$$

with $y(0)=2$ from $x=0$ to $x=4$ with various step sizes, compare the results with the true values by computing the true percent relative error $|\varepsilon_t|$.

Excel Spreadsheet can be used to solve this problem as follows:

	A	B	C	D	E	F	G	H
1	x_i	y_i	$f(x_i,y_i)$	h	k_1	$x_i+\frac{1}{4}h$	$y_i+\frac{1}{4}k_1h$	k_2
2	0	2	= 4*EXP(0.8*A2)-0.5*B2	0.5	= C2	= A2+(1/4)*D2	= B2+(1/4)*E2*D2	= 4*EXP(0.8*F2)-0.5*G2
3	= A2+D2	= U2	= 4*EXP(0.8*A3)-0.5*B3	= D2	= C3	= A3+(1/4)*D3	= B3+(1/4)*E3*D3	= 4*EXP(0.8*F3)-0.5*G3
4	---------	------	--------------------	-----	------	-------------	-----------------	------------------

	I	J	K	L	M
1	$x_i + \frac{1}{4}h$	$y_i + \frac{1}{8}k_1h + \frac{1}{8}k_2h$	k_3	$x_i + \frac{1}{2}h$	$y_i - \frac{1}{2}k_2h + k_3h$
2	= A2+(1/4)*D2	= B2+(1/8)*E2*D2+(1/8)*H2*D2	= 4*EXP(0.8*I2)-0.5*J2	= A2+(1/2)*D2	= B2-(1/2)*H2*D2+K2*D2
3	= A3+(1/4)*D3	= B3+(1/8)*E3*D3+(1/8)*H3*D3	= 4*EXP(0.8*3)-0.5*J3	= A3+(1/2)*D3	= B3-(1/2)*H3*D3+K3*D3
4	--------------	------------------------	-----------	--------------	------------------------------

	N	O	P	Q	R
1	k_4	$x_i + \frac{3}{4}h$	$y_i + \frac{3}{16}k_1h + \frac{9}{16}k_4h$	k_5	$x_i + h$
2	= 4*EXP(0.8*L2)-0.5*M2	= A2+(3/4)*D2	= B2+(3/16)*E2*D2+(9/16)*N2*D2	= 4*EXP(0.8*O2)-0.5*P2	= A2+D2
3	= 4*EXP(0.8*L3)-0.5*M3	= A3+(3/4)*D3	= B3+(3/16)*E3*D3+(9/16)*N3*D3	= 4*EXP(0.8*O3)-0.5*P3	= A3+D3
4	---------------	----------------	----------------------------	----------------	---------------

	S	T	U	V	W
1	$y_i - \frac{3}{7}k_1h + \frac{2}{7}k_2h + \frac{12}{7}k_3h - \frac{12}{7}k_4h + \frac{8}{7}k_5h$	k_6	y_{i+1}	True Value	$\lvert \varepsilon_t \rvert$
2	= B2-(3/7)*E2*D2+(2/7)*H2*D2+(12/7)*K2*D2-(12/7)*N2*D2+(8/7)*Q2*D2	= 4*EXP(0.8*R2)-0.5*S2	= B2+(1/90)*(7*E2+32*K2+12*N2+32*Q2+7*T2)*D2	= (4/1.3)*EXP(0.8*A3)-(1.4/1.3)*EXP(-0.5*A3)	= ABS((V2-U2)/V2)*100
3	= B3-(3/7)*E3*D3+(2/7)*H3*D3+(12/7)*K3*D3-(12/7)*N3*D3+(8/7)*Q3*D3	= 4*EXP(0.8*R3)-0.5*S3	= B3+(1/90)*(7*E3+32*K3+12*N3+32*Q3+7*T3)*D3	= (4/1.3)*EXP(0.8*A4)-(1.4/1.3)*EXP(-0.5*A4)	= ABS((V3-U3)/V3)*100
4	-------------------------------	-------------	----------------	-------------	-------------

The results are as follows:

	A	B	C	D	E	F	G	H
1	x_i	y_i	$f(x_i, y_i)$	h	k_1	$x_i + \frac{1}{4}h$	$y_i + \frac{1}{4}k_1h$	k_2
2	0	2	3	0.5	3	0.125	2.375	3.233184
3	0.5	3.751522	4.091538	0.5	4.091538	0.625	4.262964	4.463403
4	1	6.194633	5.804847	0.5	5.804847	1.125	6.920239	6.378293
5	1.5	9.707045	8.426945	0.5	8.426945	1.625	10.76041	9.29698
6	2	14.84393	12.39017	0.5	12.39017	2.125	16.3927	13.69944
7	2.5	22.42702	18.34271	0.5	18.34271	2.625	24.71986	20.30475
8	3	33.67718	27.25411	0.5	27.25411	3.125	37.08395	30.188
9	3.5	50.41179	40.57269	0.5	40.57269	3.625	55.48338	44.95489
10	4	75.33899	60.46063	0.5	60.46063	4.125	82.89657	67.00227

	I	J	K	L	M
1	$x_i + \frac{1}{4}h$	$y_i + {}^{\mathrm{T}}k_1h + {}^{\mathrm{T}}k_2h$	k_3	$x_i + \frac{1}{2}h$	$y_i - \frac{1}{2}k_2h + k_3h$
2	0.125	2.38957398	3.225897	0.25	2.804652423
3	0.625	4.286205875	4.451782	0.75	4.861562422
4	1.125	6.95607944	6.360373	1.25	7.780246324
5	1.625	10.8147905	9.269791	1.75	12.01769588
6	2.125	16.4745276	13.65853	2.25	18.24832992
7	2.625	24.84248825	20.24344	2.75	27.47255231
8	3.125	37.26731654	30.09632	3.25	41.17834275
9	3.625	55.75726502	44.81795	3.75	61.58204229
10	4.125	83.3054222	66.79784	4.25	91.98734573

	N	O	P	Q	R
1	k_4	$x_i + \frac{3}{4}h$	$y_i + \frac{3}{16}k_1h + \frac{9}{16}k_4h$	k_5	$x_i + h$
2	3.483285	0.375	3.260923856	3.768973	0.5
3	4.857694	0.875	5.501330181	5.304346	1
4	6.983004	1.375	8.702807524	7.66526	1.5
5	10.21195	1.875	13.36918276	11.24216	2
6	15.07442	2.375	20.24518726	16.62098	2.5
7	22.36378	2.875	30.43646374	24.6785	3
8	33.26578	3.375	45.58825832	36.7248	3.5
9	49.55113	3.875	68.15173517	54.71594	4
10	73.86273	4.375	101.7810669	81.57127	4.5

	S	T	U	V	W
1	$y_i - \frac{3}{7}k_1h + \frac{2}{7}k_2h + \frac{12}{7}k_3h - \frac{12}{7}k_4h + \frac{8}{7}k_5h$	k_6	y_{i+1}	True Value	$\lvert\varepsilon_t\rvert$
2	3.752106864	4.091245	3.751522	3.751521	2.0775E-05
3	6.195523253	5.804402	6.194633	6.194631	2.9239E-05
4	9.70838676	8.426274	9.707045	9.707042	3.3372E-05
5	14.84593931	12.38916	14.84393	14.84392	3.5547E-05
6	22.43003193	18.34121	22.42702	22.42701	3.6728E-05
7	33.68168138	27.25186	33.67718	33.67717	3.7378E-05
8	50.41850484	40.56933	50.41179	50.41177	3.7737E-05
9	75.34901101	60.45562	75.33899	75.33896	3.7936E-05
10	112.511439	90.13722	112.4965	112.4964	3.8046E-05

Now if change h from 0.5 to 0.25 we will have the results immediately as follows:

	A	B	C	D	E	F	G	H
1	x_i	y_i	$f(x_i, y_i)$	h	k_1	$x_i + \frac{1}{4}h$	$y_i + \frac{1}{4}k_1h$	k_2
2	0	2	3	0.25	3	0.0625	2.1875	3.111334
3	0.25	2.807781	3.481721	0.25	3.481721	0.3125	3.025389	3.623407
4	0.5	3.751521	4.091538	0.25	4.091538	0.5625	4.007242	4.269628
5	0.75	4.866362	4.855294	0.25	4.855294	0.8125	5.169818	5.077254
6	1	6.194631	5.804848	0.25	5.804848	1.0625	6.557434	6.07987
7	1.25	7.787509	6.979373	0.25	6.979373	1.3125	8.22372	7.318745
8	1.5	9.707042	8.426947	0.25	8.426947	1.5625	10.23373	8.844509
9	1.75	12.02861	10.20649	0.25	10.20649	1.8125	12.66652	10.7192
10	2	14.84392	12.39017	0.25	12.39017	2.0625	15.61831	13.01877
11	2.25	18.26467	15.06625	0.25	15.06625	2.3125	19.20632	15.83612
12	2.5	22.42701	18.34272	0.25	18.34272	2.5625	23.57343	19.28489
13	2.75	27.49698	22.35156	0.25	22.35156	2.8125	28.89396	23.50397
14	3	33.67717	27.25412	0.25	27.25412	3.0625	35.38055	28.66311
15	3.25	41.21483	33.24754	0.25	33.24754	3.3125	43.2928	34.96976
16	3.5	50.41177	40.5727	0.25	40.5727	3.5625	52.94757	42.67734
17	3.75	61.6365	49.5239	0.25	49.5239	3.8125	64.73174	52.09551
18	4	75.33896	60.46064	0.25	60.46064	4.0625	79.11775	63.60248

	I	J	K	L	M
1	$x_i + {}^{T}h$	$y_i + {}^{T}k_1h + {}^{T}k_2h$	k_3	$x_i + \frac{1}{2}h$	$y_i - \frac{1}{2}k_2h + k_3h$
2	0.0625	2.1909792	3.109595	0.125	2.388481898
3	0.3125	3.02981631	3.621194	0.375	3.260153521
4	0.5625	4.012807753	4.266845	0.625	4.284529104
5	0.8125	5.17675389	5.073786	0.875	5.500151519
6	1.0625	6.566028875	6.075573	1.125	6.9535409
7	1.3125	8.234324952	7.313442	1.375	8.701026191
8	1.5625	10.24677501	8.837984	1.625	10.81097452
9	1.8125	12.68253812	10.71119	1.875	13.3665074
10	2.0625	15.63795126	13.00894	2.125	16.4688123
11	2.3125	19.23037352	15.82409	2.375	20.24118212
12	2.5625	23.6028765	19.27017	2.625	24.83394444
13	2.8125	28.92996904	23.48596	2.875	30.43047788
14	3.0625	35.42458555	28.64109	3.125	37.25455697
15	3.3125	43.34661841	34.94285	3.375	45.57931996
16	3.5625	53.01333644	42.64446	3.625	55.7382193
17	3.8125	64.81210759	52.05532	3.875	68.13839408
18	4.0625	79.21593602	63.55339	4.125	83.277001

	N	O	P	Q	R
1	k_4	$x_i+\frac{3}{4}h$	$y_i+\frac{3}{16}k_1h+\frac{9}{16}k_4h$	k_5	x_i+h
2	3.226443	0.1875	2.594343508	3.350165	0.25
3	3.769358	0.4375	3.501052747	3.925744	0.5
4	4.452621	0.6875	4.569461939	4.648281	0.75
5	5.304935	0.9375	5.839960155	5.54802	1
6	6.361642	1.1875	7.361339587	6.662169	1.25
7	7.666151	1.4375	9.192719367	8.036412	1.5
8	9.271699	1.6875	11.40588789	9.726758	1.75
9	11.2435	1.9375	14.08815715	11.8018	2
10	13.66138	2.1875	17.34584326	14.34549	2.25
11	16.62299	2.4375	21.30851245	17.46049	2.5
12	20.24771	2.6875	26.13416258	21.27235	2.75
13	24.68149	2.9375	32.01554794	25.9345	3
14	30.1027	3.1875	39.18790081	31.63446	3.25
15	36.72927	3.4375	47.93835951	38.60135	3.5
16	44.82747	3.6875	58.61748111	47.11507	3.75
17	54.72261	3.9375	71.6533007	57.51761	4
18	66.81206	4.1875	87.56850118	70.22668	4.25

	S	T	U	V	W
1	$y_i-\frac{3}{7}k_1h+\frac{2}{7}k_2h+\frac{12}{7}k_3h-\frac{12}{7}k_4h+\frac{8}{7}k_5h$	k_6	y_{i+1}	True Value	$\lvert\varepsilon_t\rvert$
2	2.807921973	3.48165	2.807781	2.807781	3.9287E-07
3	3.751696218	4.091451	3.751521	3.751521	6.2597E-07
4	4.866577808	4.855186	4.866362	4.866362	7.7594E-07
5	6.194897507	5.804715	6.194631	6.194631	8.7731E-07
6	7.787835676	6.979209	7.787509	7.787509	9.4802E-07
7	9.707442997	8.426746	9.707042	9.707042	9.9838E-07
8	12.02910138	10.20625	12.02861	12.02861	1.0347E-06
9	14.84452337	12.38987	14.84392	14.84392	1.0613E-06
10	18.26540995	15.06588	18.26467	18.26467	1.0807E-06
11	22.42791333	18.34227	22.42701	22.42701	1.095E-06
12	27.49808335	22.35101	27.49698	27.49698	1.1056E-06
13	33.67851591	27.25345	33.67717	33.67717	1.1135E-06
14	41.21646998	33.24672	41.21483	41.21483	1.1193E-06
15	50.41377869	40.5717	50.41177	50.41177	1.1236E-06
16	61.63895213	49.52267	61.6365	61.6365	1.1267E-06
17	75.34195744	60.45914	75.33896	75.33896	1.1291E-06
18	92.07226912	73.82027	92.06861	92.06861	1.1309E-06

Conclusion

Using of Excel Spreadsheet makes the computation very simple and we obtained a very good results which are very close to the true values and also can use any value for the step size and will obtain the results quickly.

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