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Critical Temperature of Short Cylindrical Shells Based on Improved Stability Equation

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Abstract: The nonlinear strain-displacement relations in general cylindrical coordinates are simplified by Sander's assumptions for the cylindrical shells and substituted into the total potential energy function for thermoelastic loading. The Euler equations are then applied to the functional of energy, and the general thermoelastic equations of nonlinear shell theory are obtained and compared with the Donnel equations. An improvement is observed in the resulting equations as no length limitations are imposed on a thin cylindrical shell. The stability equations are then derived through the second variation of potential energy, and the same improvements are extended to the resulting thermoelastic stability equations. Based on the improved equilibrium and stability equations, the magnitude of thermoelastic buckling of thin cylindrical shells under different thermal loading is obtained. The results are extended to short and long thin cylindrical shells.

Key words: Stability, Critical Temperature, Thermoelastic Buckling, Cylindrical Shells

Introduction

The theoretical developments of the elastic buckling of shells may be attributed to the works of Donnel (1950; 1958), who evaluated the critical buckling load of a short cylindrical shell that is frequently used in industry. The prime importance of his formulation consists of simplicity and the closed-from solutions that he obtained for critical loads for some loading conditions that frequently occur in practical design problems. These formulations include the effect of imperfections and present the analysis for a short cylindrical shell under axial compression and external pressure while ignoring the shear force in circumforential direction and the rotations β_x and β_θ. These assumptions lead to results that are acceptable for short cylindrical shells where the transverse shear force and rotations are small and can be ignored. The equilibrium and stability equations that he obtained were essentially based on the force summation method and have become the basis of many other developments for the shell buckling theory such as treatments presented by Donnel (1976), Koiter (1977; 1980), Flugge (1973) Morley (1959), and Brush and Almroth (1975). The analyses presented in these papers are restricted to the mechanical loads such as axial compression, lateral pressure, twisting moments, or combinations thereof. The method of solution is generally based on trigonometric approximation for the displacement components and substitution into the stability equations and setting the determinant of coefficients to zero. The prebuckling stresses are the associated membrane stresses for each individual loading. Thermal buckling of cylindrical shells based on Donnel equations are given by Johns (1962), Chang and Cord(1970) and for the shell of revolution by Bushnell (1971). Thermal buckling of short cylindrical shells fixed between two disks that are allowed to move in the direction of shell axis, where the disks are kept at low temperatures and the shell at a high temperature, is considered by Lukasiewicz (1980). In this article the improved equilibrium and stability equations are obtained and employed to compute the critical thermoelastic buckling load of cylindrical shells under radial thermal loadings, axial temperature difference, and critical uniform final temperature for simply supported shells.

Analysis: A thin cylindrical shell of mean radius R and thickness h with L is considered. The normal and shear strains at a distance z from the middle Planes of the shell are (Donnel, 1976):

$$\varepsilon_x = \varepsilon_{xm} + zk_x$$

$$\varepsilon_\theta = \varepsilon_{\theta m} + zk_\theta$$

$$\gamma_{x\theta} = \gamma_{x\theta m} + zk_{x\theta} \qquad (1)$$

Where the $\varepsilon' S$ are the normal strains, $\gamma' S$ are the shear strains, and k_{ij} are the curvatures. The subscript m refers to the strain at the middle surface of the shell. The indices x and θ refer to the axial and circumferential directions, respectively. According to the Sander's assumption (Sanders, 1963), the general strain-displacement relations can be simplified to given the following terms for the strains at the middle surface and the curvatures in terms of displacement components:

$$\varepsilon_{xm} = u_{,x} + 0.5w^2_{,x}$$

$$\varepsilon_{\theta m} = (v_{,\theta} + w)/R + (v - w_{,\theta})^2/2R^2$$

$$\gamma_{x\theta m} = u_{,\theta}/R + v_{,x} + (-w_{,x}v + w_{,x}w_{,\theta})/R$$

$$k_x = -w_{,xx}$$

$$k_\theta = (v_{,\theta} - w_{,\theta\theta})/R^2$$

$$k_{x\theta} = (v_{,x} - 2w_{,x\theta})/2R \qquad (2)$$

where u ,v and w are the displacements and (,) indicates a partial derivative. The Hook's law in terms of forces and moments per unit length is

$$N_x = C(\varepsilon_{xm} + \nu\varepsilon_{\theta m}) - E\alpha T_0/(1-\nu)$$
$$N_\theta = C(\varepsilon_{\theta m} + \nu\varepsilon_{xm}) - E\alpha T_0/(1-\nu)$$
$$N_{x\theta} = N_{\theta x} = C(1-\nu)\gamma_{x\theta m}/2$$
$$M_x = D(k_x = \nu k_\theta) - E\alpha T_1/(1-\nu)$$
$$M_\theta = D(k_\theta + \nu k_x) - E\alpha T_1/(1-\nu)$$
$$M_{x\theta} = M_{\theta x} = D(1-\nu)k_{x\theta} \quad (3)$$

Where N_{ij} and M_{ij} are related to σ_{ij} through the shell thickness according to the first-order shell theory is the elastic modules, v is the Poisson's ratio α is the coefficient of thermal expansion. Also:

$$T_0 = \int_{-h/2}^{h/2} T dz, \; T_1 = \int_{-h/2}^{h/2} Tz dz$$
0and
$$C = Eh/(1-\nu^2), D = Eh^3/12(1-\nu^2) \quad (4)$$

The total potential energy of the, shell is the sum of membrane strain energy U_m, the bending strain energy U_b, and the thermal strain energy U_T, expressed as:

$$U = U_m + U_b + U_T \quad (5)$$
where
$$U_m = RC/2\iint[\varepsilon^2_{xm} + \varepsilon^2_{\theta m} + 2\nu\varepsilon_{xm}\varepsilon_{\theta m} + (1-\nu)\gamma^2_{x\theta m}/2]dxd\theta$$
$$U_b = RD/2\iint[K^2_x + K^2_\theta + 2\nu K_x K_\theta + 2(1-\nu)K^2_{x\theta}]dxd\theta$$
$$U_T = -RE\alpha/(1-\nu)\iint[(\varepsilon_{xm} + \varepsilon_{\theta m})T_0 + (K_x + K_\theta)T_1]dxd\theta + RE\alpha^2/(1-\nu)\iiint T^2 dxd\theta dz \quad (6)$$

Assuming that the cylindrical shell is under thermal stress alone, the total potential energy is a function of the displacement components and their derivatives and can be written as:

$$U = \iiint F(u, v, w, u_{,x}, u_{,\theta}, v_{,x}, v_{,\theta}, w_{,x}, w_{,\theta}, w_{,xx}, w_{,\theta\theta}, w_{,x\theta})dxd\theta dz$$

Minimizing the functional of potential energy leads to the Euler equations:

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial u_{,\theta}} = 0$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial v_{,\theta}} = 0$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w_{,x}} - \frac{\partial}{\partial \theta}\frac{\partial F}{\partial w_{,\theta}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{,xx}} + \frac{\partial^2}{\partial x\partial\theta}\frac{\partial F}{\partial w_{,x\theta}} + \frac{\partial^2}{\partial \theta^2}\frac{\partial F}{\partial w_{,\theta\theta}} = 0 \quad (7)$$

Upon substitution from Eqs. (6) into (7) and using Eqs. (2) and (3),the equilibrium equations for general thin cylindrical shell are obtained as:

$$RN_{x,x} + N_{x\theta,\theta} = 0$$
$$RN_{x\theta m x} + N_{\theta,\theta} + 1/RM_{\theta,\theta} + M_{x\theta,x} - (N_\theta\beta_\theta + N_{x\theta}\beta_x) = 0 \quad (8)$$
$$RM_{x,xx} + 2RM_{x\theta,x\theta} + M_{\theta,\theta\theta} - RN_\theta - R[RN_x\beta_{x,x} + N_{x\theta}(R\beta_{\theta x} + \beta_{x,\theta})]N_\theta\beta_{\theta,\theta} = 0$$

Where

$$\beta_x = w_{,x} \quad \text{and} \quad \beta_\theta = (\nu - w_{,\theta})/R \quad (9)$$

are the rotation in the x-and θ directions. It is to be noted that the first and the third equilibrium equations (8) are identical to the Donnel equations. The second equilibrium equation (8) has extra terms compared to the Donnel equations. Here we recall that the transverse shear force in the circumferential direction is:

$$Q_\theta = M_{\theta,\theta}/R + M_{x\theta,x} \quad (10)$$

Comparison shows that in the Donnel equations for short cylindrical shells the shear force in the circumferential direction and rotations β_x and β_θ are ignored. These approximations are justified for short cylinders these assumptions are no longer sufficient and the inclusion of these terms adds to the accuracy of the results.

The stability equations of thin cylindrical shells can be derived by the variational formulations. If V is total potential energy of the shell, the expantion of V about the equilbrium state into the Taylor series yields:

$$\Delta V = \delta V + \frac{1}{2}\delta^2 V + \frac{1}{3!}\delta^3 V + \ldots \quad (11)$$

The first variation δV is associated with the state of equilbrium. The stability of the original configuration of the shell in the neighbohood of equilbrium state can determined by the sign of second variation $\delta^2 V$ as follows:

- The equilbrium is stabble if $\delta^2 V > 0$ for all virtual displacements.
- The equilbrium is unstable if $\delta^2 V < 0$ for at least one admissible set of virtual displacements.
- The condition $\delta^2 V = 0$ is used to drive the stability equations for many practical buckling problems.

The external load acting on the original configuration is considered to be the critical buckling load if the following variational equation is satisfied :

$$\delta(\delta^2 V) = 0 \quad (12)$$

This rule provides the govering equations that determine the lowest critical load.

Consider the state of stable equilbrium of a general cylindrical shell under thermal load that is designated by u_0, v_0 and w_0. The displacement components of the neighboring state are:

$$u = u_0 + u_1$$

$$u = u_0 + u_1$$

$$w = w_0 + w_1 \qquad (13)$$

Similarly, the components of forces and moments related to the neighboring state are related to the state of equilibrium according to the relations:

$$N_x = N_{x0} + \Delta N_x \qquad M_x = M_{x0} + \Delta M_x$$
$$N_\theta = N_{\theta 0} + \Delta N_\theta \qquad M_\theta = M_{\theta 0} + \Delta M_\theta$$
$$N_{x\theta} = N_{x\theta 0} + \Delta N_{x\theta} \qquad M_{x\theta} = M_{x\theta 0} + \Delta M_{x\theta} \qquad (14)$$

If $\Delta N_x, \Delta N_\theta, \Delta_{x\theta}, \ldots$ express the linear portion of In terms of u_1, v_1, and w_1 they become:

$$N_{x1} = C(\varepsilon_{x1} + v\varepsilon_{\theta 1}) \qquad M_{x1} = D(K_{x1} + vK_{\theta 1})$$
$$N_{\theta 1} = C(\varepsilon_{\theta 1} + v\varepsilon_{x1}) \qquad M_{\theta 1} = D(K_{\theta 1} + vK_{x1})$$
$$N_{x\theta 1} = N_{\theta x1} = \frac{C}{2}(1-v)\gamma_{x\theta 1} \qquad M_{x\theta 1} = M_{\theta x1} = D(1-v)K_{x\theta 1} \qquad (15)$$

and

$$N_{x0} = C(\varepsilon_{x0} + v\varepsilon_{\theta 0}) - E\alpha T_0/(1-v)$$
$$N_{\theta 0} = C(\varepsilon_{\theta 0} + v\varepsilon_{x0}) - E\alpha T_0/(1-v)$$
$$N_{x\theta 0} = \frac{C}{2}(1-v)\gamma_{x\theta 0}$$
$$M_{x0} = D(K_{x0} + vK_{\theta 0}) - E\alpha T_1/(1-v)$$
$$M_{\theta 0} = D(K_{\theta 0} + vK_{x0}) - E\alpha T_1/(1-v)$$
$$M_{x\theta 0} = D(1-v)K_{x\theta 0} \qquad (16)$$

Using Eqs. (13) for the displacement components of a neighboring state of stable equilibrium and employing the decomposed linear part of strain displacement relations, the linear strain relations for the equilibrium state (0) and the first variation (1) are obtained as:

$$e_x = e_{x0} + e_{x1} \qquad \beta = \beta_{\theta 0} + \beta_{\theta 1}$$
$$e_\theta = e_{\theta 0} + e_{\theta 1} \qquad K_x = K_{x0} + K_{x1}$$
$$e_{x\theta} = e_{x\theta 0} + e_{x\theta 1} \qquad K_\theta = K_{\theta 0} + K_{\theta 1}$$
$$\beta_x = \beta_{x0} + \beta_{x1} \qquad K_{x\theta} = K_{x\theta 0} + K_{x\theta 1} \qquad (17)$$

Substituting for the strains and curvatures from the linearize strain-displacement relations yields:

$$\varepsilon_{x1} = e_{x1} = u_{1,x} \qquad K_{x1} = -w_{1,xx}$$
$$\varepsilon_{\theta 1} = e_{\theta 1} = (v_{1,\theta} + w_1)/R \qquad K_{\theta 1} = (v_{1,\theta} - w_{1,\theta\theta})/R^2$$
$$\gamma_{x\theta 1} = e_{x\theta 1} = v_{1,x} + u_{1,\theta}/R \qquad K_{x\theta 1} = (v_{1,x} - 2w_{1,x\theta})/R$$
$$\beta_{x1} = -w_{1,x} \qquad \beta_{\theta 1} = (v_1 - w_{1,\theta})/R \qquad (18)$$

Equations (18) are written in terms of the displacement components, which upon application of Euler equations result in the stability equations:

$$RN_{x1,x} + N_{x\theta 1,\theta} = 0$$
$$RN_{x\theta 1,x} + N_{\theta 1,\theta} + 1/RM_{\theta 1,\theta} + M_{x\theta 1,x} - (N_{\theta 0}\beta_{\theta 1} + N_{x\theta 0}\beta_{x1}) = 0$$
$$RM_{x1,xx} + 2M_{x\theta 1,x\theta} + M_{\theta 1,\theta\theta}/R - N_{\theta 1}$$
$$-[RN_{x0}\beta_{x1,x} + N_{x\theta 0}(R\beta_{\theta 1,x} + \beta_{x1,\theta}) + N_{\theta 0}\beta_{\theta 1,\theta} = 0 \qquad (19)$$

A comparison of these equations with the Donnel stability equations reveals that while the first and third equations are identical, the second equation includes extra terms that are ignored in the Donnel equations because they are small for short cylinders. Including these terms removes the length limitations imposed by Donnel equations. It should be noted that these terms improve the accuracy of the predicted design critical loads.

Thermal Buckling, Short Cylindrical Shells: For short cylindrical shells the transverses shear force $O\theta$ and rotations β_x and $\beta\theta$ are ignored and the equilibrium and stability equations reduce to the Donnel equations. Upon substitution for N_{ij} and M_{ij} their equivalencies of strain from Eqs. (15) and finally in terms of displacements from Eqs. (18), the uncoupled forms of the Donnel equations are obtained as:

$$\nabla^4 u_1 = -v/Rw_{1,xxx} + 1/R^3 w_{1,x\theta\theta}$$
$$\nabla^4 r_1 = -1/R^4 w_{1,\theta\theta\theta} - (2+v)/R^2 w_{1,xx\theta}$$
$$D\nabla^8 w_1 + C(1-v^2)/R^2 w_{1,xxxx}$$
$$-\nabla^4[N_{x0}w_{1,xx} + 2/RN_{x\theta 0}w_{1,x\theta} + N_{\theta 0}w_{1,\theta\theta}/R^2] = 0 \qquad (20)$$

These equations are related to the thermal stresses through the prebuckling terms such as N_{x0} through Eqs. (16) In the next section three types of thermal bucklings are discussed and the critical temperatures are calculated.

Critical Initial - Final Temperature: Consider a cylindrical shell of length L, radius R, and thickness h with both ends simply supported. The initial uniform temperature of the shell is assumed to be T_p if the shell is simply supported and the axial displacement is prevented, the temperature can be uniformly raised to a final value T_1 such that the shell buckles. To find the critical $\Delta T = T_1 - T_i$, the prebuckling thermal stresses are:

$$N_{x0} = -E\alpha/(1-v)\int_{-h/2}^{h/2} T dz = -\beta\Delta T \qquad (21)$$

where

$\beta = E\alpha h/(1-v)$. For this type of loading $N_{\theta 0} = N_{x\theta 0} = 0$ .from the edge conditions:

$$w_1 = w_{1,xx} = 0 \qquad (22)$$

Assume the solution in the form:

$$w_1 = C_1 \sin\bar{m}x \sin n\theta \quad (23)$$

where $\bar{m} = m\pi R/L$ and C_1 is a constant. Coefficient.

Substituting Eqs.(21) into third of Eqs. (20) yields:

$$D\nabla^8 w_1 + C(1-v^2)/R^2 w_{1,xxxx} + \beta\Delta T\nabla^4 w_{1,xx} = 0 \quad (24)$$

Substituting the solution w from Eq .(23) yields:

$$\Delta T = \frac{D(\bar{m}^2 + n^2/R^2)^2}{\bar{m}^2} + \left(\frac{1-v^2}{R^2}\right)\frac{C\bar{m}^2}{(\bar{m}^2 + n^2/R^2)^2} \quad (25)$$

The critical temperature depends upon m and n. Denoting $\gamma = (\bar{m}^2 + n^2/R^2)^2/\bar{m}$ Eq. (25) reduces to:

$$\beta\Delta T = D\gamma + c(1v^2)/R^2\gamma \quad (26)$$

Minimizing ΔT with respect to γ gives:

$$\gamma = [c(1-v^2)/DR^2]^{1/2} \quad (27)$$

where C and are constants defined in Eqs. (3) Substituting into Eq. (26) yield the critical temperature difference as:

$$\Delta T_{crit} = \frac{Eh^2/R}{\beta[3(1-v^2)]^{1.2}} = \frac{h}{R\alpha}\left[\frac{1-v}{3(1+v)}\right]^{1.2} \quad (28)$$

For v= 0.3 we have

$$\Delta T_{crit} = 0.424h/R\alpha \quad (29)$$

This value for critical temperature can be compared with the relation given b Johns (1962). considered a cylindrical shell stiffened by solid rings under uniform temperature. The critical initial – final temperature that he obtained is

$$\Delta T_{crit} = kh/R\alpha \quad (30)$$

For simply supported cylindrical shells, k = 5.3 Johns, 1962). Comparing the factor k = 5 given by Johns with 0.424 of Eq. (29) Shows that the critical limit proposed John is 12 times higher. This factor is reasonable due to the Johns assumption.

It is further noticed that critical temperature for short cylindrical shells does not depend upon the shell length.

Critical Radial Temperature: Assume linear temperature Variation across the shell thickness as:

$$\Delta T(z) = \Delta T(z + h/2)/h \quad (31)$$

Where z measures from the middle plane of the shell. For simply supported edges the prebuckling axial force in the shell is:

$$N_{x\theta} = -E\alpha h\Delta T/2(1-v) \quad (32)$$

With a similar method, the critical temperature difference between inside and outside surfaces is:

$$\Delta T_{crit} = (T_1 - T_o)_{crit} = [4(1-v)/3(1+v)]^{1/2} h/R\alpha \quad (33)$$

For v = 0.3:

$$\Delta T_{crit} = 0.848h/R\alpha \quad (34)$$

Critical Axial Temperature: Consider a cylindrical shell of length L under axial temperature difference and with simply supported edges, where the motion of the edges in the axial direction is prevented. Assume a linear temperature variation in the axial direction x:

$$T(x) = \Delta Tx/L \quad (35)$$

Where

$\Delta T = T(L) - T(0)$. The prebuckling axial force is:

$$N_{x0} = -E\alpha h/(1-v)\Delta TxL = -\beta\Delta Tx/L \quad (36)$$

Substituting this into the third of the stability equations gives:

$$D\nabla^R w_1 + Eh/R^2 w_{1.xxxx} + \beta\Delta Tx/L\nabla^4 w_{1.xx} = 0 \quad (37)$$

Expanding T (x) from Eq. (35) into a Fourier series, keeping only the first two terms, assuming a series

solution for w_1 with two terms as:

$$w_1 = \sum_{m-1}^{2} a_m \sin n\theta \sin\bar{m}x \quad (38)$$

and substituting Eq. (38) into Eq. (37) results in two

equations for a_1 and a_2 that should be solved simultaneously if Eq. (37) is to be satisfied. The

determinant of the resulting two equations for a_1 and a; yields the following relation for the critical axial temperature difference:

$$\Delta T_{crit} = \frac{(1/4)(K_{1n}P_{2n} + K_{2n}P_{1n}) \pm \sqrt{\left[\frac{1}{4}(K_{1n}P_{2n} + P_{1n}K_{2n})^2 - .22P_{1n}P_{2n}K_{1n}K_{2n}\right]}}{.22\beta P_{1n}P_{2n}} \quad (39)$$

where:

$$k_{mn} = D(\bar{m}^2 + n^2/k^2)^4 + Eh\bar{m}^4/R^2$$

$$p_{mn} = (\bar{m}^2 + n^2/R^2)^2\bar{m}^2$$

$$\bar{m} = m\pi R/L \quad (40)$$

Equation (39) is used to obtain the critical temperature. For a cylindrical shell with R = 1000 mm, h = 5 mm, $\alpha = 11.9 \times 10^{-6}$ 1/ °C, and v = 0.3, the critical

temperatures for two values of L/R are given in Table 1. The value of m in Eq. (39) is either 1 or 2; and the values of n, the associated circumferential waves at which the buckling appears, are also given in Table 1.
It is noted that when L/R is increased, the number of circumferential waves n is decreased. Also, when L/R is increased, the critical axial temperature is also increased.

Table 1: Critical Temperature for Axial Loading

L/R	n	ΔT_{crit}
0.5	11	270.48
10	3	320.79

Conclusion

The theory of thermal buckling of cylindrical shells has many applications in industrial design problems. Frequently, the thermal loading encountered in buckling design problems of cylindrical shells include simple temperature distributions between inside and outside surfaces, initial final temperature differences, and the axial temperature distributions are commonly practiced in design problems. The simple design formula presented in this paper should provide an effective means for analysis of practical problems of thermal buckling of cylindrical shells.

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