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A Comprehensive Overview and Latest Studies on the Green's Function of Some Differential Equations with Operator Coefficient

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Abstract: Forming of the Green's Function of some differential equations with operator coefficient has been overviewed in this study. Various examples studied in spectral analysis have been given and the results have been discussed.

Key Words: Spectrum, Green function, Operator

Introduction

Various equations of mathematical physics, such as partial differential equations, integro-differential equations, infinite number of ordinary differential equations can be written as ordinary differential equation with operator coefficient. Therefore, it is important to study Green's Function of this kind of equations. There have been many researches on Green's Function of differential equations with different operator coefficient since last 35 years. Spectral analysis of Sturm-Liouville equation with unbounded operator coefficient was firstly given by Kostyuchenko, and Levitan, 1967.

Many journal papers and books have been published on the spectral analysis of ordinary differential equations with self-adjoint and/or normal operator coefficient. For example, the books of Titchmarsh, (1962), Coddington and Levinson, (1955), Naimark, (1969), Levitan, and Sargsyan, (1991) can be shown as main references. The work of Fulton, and Pruess, (1994) is also one of the latest studies and this work has many references about the subject.

Let H is separable Hilbert space. Let us represent the set of functions by different Halberd spaces which is defined in the interval $0<x<\infty$ whose values belong to H, strongly measurable and satisfying the condition

$$\int_0^\infty \|f(x)\|_H^2 dx < \infty .$$

The inner product of f(x) and g(x) functions belonging to H_1 is defined by the formula

$$(f,g)_1 = \int_0^\infty (f(x), g(x))_H dx \cdot$$

H_1 forms a separable Hilbert space (Balakrishnan, 1976). Here, $\|.\|$, $(.,.)$ show the norm and inner product in H, respectively.

Green's Function of Sturm-Liouville equation with unbounded operator coefficient has been investigated by Levitan, (1968) at first. In 1971, forming of Green's Function of 2nth order differential expression such that

$$(-1)^n y^{(2n)} + \sum_{j=1}^{2n} Q_j(x) y^{(2n-j)} \quad , \qquad -\infty < x < \infty$$

and asymptotic expression of eigenvalues has been given in the work of Bayramoglu (1971). Here, $Q_j(x)$ (j=2,...,2n) are the self-adjoint operators in space H. Later on, many other papers have been published on this subject (Aslanov, 1976, 1994; Boymatov, 1973; Kasımov, 1980; Misnayevsky, 1976; Kleyman, 1977; Otelbayev, 1977; Oer, 1997; Saito, 1975; Tamura, 1974 and Solomyak, 1986). References of the studies by 1990 have been given in the book of Otelbayev, (1990). The book of Stakgold, (1979) can be shown as a reference for investigation of Green's Function of ordinary differential equations.

Separation problem for the scalar Sturm-Liouville problem has been investigated by English mathematicians, W.N. Everitt and Giertz (1973, 1977) at first. Separation problem for the operator formed by differential expressions with operator coefficient has been studied by M. Otelbayev and their students in Alma Ata, K. Boymatov and their students in Dusenbe, M. Gasımov, F. Maksudov, M. Bayramoglu, and their students in Baku. Furthermore, there have been some studies about this problem in different science centres.

Although there has been many studies on the spectral analysis of differential equations with operator coefficient, there are still many problems which needs to be studied.

Green's Function of Differential Equations with Self-adjoint Operator Coefficient: Considering the set of functions by $H_1 = L_2(a,b;H)$ which is defined in the interval a<x<b $(-\infty \le a < x < b \le \infty)$ whose values belong to H, strongly measurable.

In (Oer, 1997), Green function of L operator formed by Sturm-Liouville differential expression in space

$$H_1 = L_2(0,\infty;H)$$

$$-y'' + Q(x)y$$

and the boundary condition

$$y'(0) - hy(0) = 0$$

where h is complex number has been investigated.

Then, the separation theorem for the operator formed by the expression

$$-y'' + Q(x)y$$

in space $L_2(-\infty,\infty;H)$ has been proved.

Here, Q(x) is an operator, which transforms at H in each value of x, self-adjoint, lower bounded and inverse of it is completely continuous.

Example 1: Consider the simple and unlocal Schrödinger operator with reciprocal effect met in Quantum mechanics (Dolph, 1961).

$$A\psi = -\Delta\psi + \int v(r)v(r')\psi(r')dr'$$

$$r=(x,y,z) \qquad r'=(x',y',z')$$

Let it write the boundary value problem corresponding to A.

$$-\Delta U + \iiint K(x,x')U(x')dx' = \lambda U \qquad (1)$$

$$U\big|_c = 0 \ , \ \left[\frac{\partial U}{\partial x_1} - hU\right]\Bigg|_{x_1=0} \qquad (2)$$

$$x=(x_1,x_2,x_3) \qquad x'=(x_1',x_2',x_3')$$

where base s is a cylinder with plane region $\underline{o}$:

$$(x_2,x_3)\in \underline{o} \qquad 0\le x_1<\infty ,$$

where c is border of the cylinder. Assumed that the kernel K(x,x')

$$K(x,x')=\overline{K(x',x)}$$

and

$$\int_s \int_s |K(x,x')|^2 dxdx' < \infty \quad (dx = dx_1dx_2dx_3 ,\ dx' = dx_1'dx_2'dx_3')$$

Therefore, the problem (1), (2) is written such as

$$-\frac{\partial^2 U}{\partial x_1^2} + Q(x_1)U = \lambda U \qquad (3)$$

$$U'(0) - hU(0) = 0 \qquad (4)$$

where

$$Q(x_1)U = -\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial^2 U}{\partial x_3^2} + \iiint K(x,x')U(x')dx' .$$

Thus, the Green function of the problem (1), (2) is reduced the examining of the Green function of the problem (3), (4).

Example 2: Let consider the expression $-y''+Q(x)y$ in space $L_2(1,\infty)$ where

$$Q(x) = -4x^2 + \frac{2}{x^2} - 2\frac{\cos x^2}{\sin x^2} \qquad -y''+Q(x)y$$

Let $y=\frac{\sin x^2}{x}$. $\frac{\sin x^2}{x}\in L_2(1,\infty)$. Although

$$-y''+Q(x)y = 0 \in L_2(1,\infty)$$

$$y'' = -4x\sin x^2 - 2\frac{\cos x^2}{x} + 2\frac{\sin x^2}{x^3} \notin L_2(1,\infty) .$$

Thus the differential operator

$$-y'' - \left(4x^2 - \frac{2}{x^2} + 2\frac{\cos x^2}{\sin x^2}\right)y \qquad 1\le x<\infty$$

is not separable.

Example 3: Let Ω is regular bounded any region in plane xy. Let us consider the differential expression

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + q(x,y,z)u \qquad (5)$$

in cylinder $T=\Omega\times(-\infty,\infty)$ of the three dimensional space. Here, q(x,y,z) is a positive valued continuous function defined in cylinder T. In addition, assumed that satisfies the condition

$$\left|(q(x,y,z) - q(x,y,\xi))q^{-1}(x,y,z)\right| \le c$$

while $|x-\xi|\le 2$.

Let define the expression (5) in $L_2(T)$ and its domain set satisfying the condition

$$u\big|_{(x,y,z)\in\partial\Omega} = 0 .$$

Let us write the operator formed above as a differential expression with operator coefficient.

Here, $\partial\Omega$ is border of Ω.

$$-\frac{d^2 u}{dz^2} + Q(z)u$$

Let Q(z) is an operator formed by the expression

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + q(x,y,z)u$$

for every z in $(-\infty,\infty)$ in space $H=L_2(\Omega)$ and the condition

$$u\big|_{(x,y,z)\in\partial\Omega} = 0 .$$

Operator Q(z) is a self-adjoint operator in $L_2(\Omega)$ and inverse of it is a completely continuous operator. Since q(x,y,z) is positive, operator Q(z) is positive. Q(z) satisfies the following conditions (Kostyuchenko and Levitan, 1967).

1) D, domain set of operators $Q(x)=Q^*(x)$ is independent from x and let $\overline{D}=H$.
2) Let $(Q(x)f,f)\ge(f,f)$ for $f\in D$ and $Q^{-1}(x)$ is completely continuous in H.
3) Let $\left\|[Q(s)-Q(x)]Q^{-a}(x)\right\| \le c|x-s|$, $0<a<\frac{3}{2}$ while $|x-s|\le 1$ (c is a positive constant).

According to this, if $u\in L_2(T)$ and

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + q(x,y,z)u \in L_2(T)$$

then

$\frac{\partial^2 u}{\partial x^2} \in L_2(T)$ and $-\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + q(x,y,z)u \in L_2(T)$.

It is possible to give more examples.

Example 4: Let $H = R^3$ is a three-dimensional Euclidean space and

$$Q(x) = \begin{pmatrix} |x|+2 & 1 & 1 \\ 1 & |x|+2 & 1 \\ 1 & 1 & |x|+2 \end{pmatrix}.$$

It is seen that Q(x) satisfies the conditions 1)-3). Therefore, the operator formed by

$$-y'' + Q(x)y$$

is separable in the space $L_2(-\infty,\infty;R^3)$.

In another work (Albayrak, 1997), considering Hilbert space $L_2(0,\infty,H)$, Green function of boundary value problem defined by

$$y^{IV} + Q(x)y, \qquad 0 < x < \infty$$

and

$$y'(0) - h_1 y(0) = 0$$

$$y'''(0) - h_1 y''(0) = 0$$

was studied. Here, Q(x) is a self-adjoint operator and bounded below for every $x \in [0,\infty)$ and its inverse is compact in H, h_1 is an arbitrary complex number.

Example 5: Let $H = C^n$, where C^n is an n-dimensional complex Euclidean space. Let $Q(x) = (q_{ij}(x))_{i,j=1}^n$. Hence, the problem examined is reduced to investigating the Green function of boundary value problem

$$y_i^{IV} + \sum_{j=1}^{n} q_{ij}(x) y_i, \qquad 0 < x < \infty$$

$$y_i'(0) - h_1 y_i(0) = 0$$

$$y_i'''(0) - h_1 y_i''(0) = 0, \quad i=1,2,3, \quad ,n$$

in space $L_2(0,\infty,C^n)$. Here

$q_{ij} = \overline{q}_{ji}(x)$ are ordinary differential functions.

Example 6: Problem examined in case $H = l_2$ can be reduced investigating Green function of the following infinite system in $L_2(0,\infty,l_2)$:

$$y_i^{IV} + \sum_{j=1}^{\infty} q_{ij}(x) y_i$$

$$y_i'(0) - h_1 y_i(0) = 0$$

$$y_i'''(0) - h_1 y_i''(0) = 0,$$

where ordinary functions q_{ij} are functions satisfying the condition $q_{ij} = \overline{q}_{ji}(x)$.

Example 7: Let consider the following boundary value problem:

$$\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + q(x,y)u, \qquad 0 < x < \infty, \qquad 0 < y < \pi \quad (6)$$

$$u_x(0,y) - h_1 u(0,y) = 0 \qquad 0 < y < \pi \quad (7)$$

$$u_{xxx}(0,y) - h_1 u_x(0,y) = 0 \qquad 0 < y < \pi \quad (8)$$

$$u(x,0) = u(x,\pi) = 0. \quad (9)$$

Here q(x,y) is a real continuous function in $[0,\infty) \times [0,\pi]$, h_1 is a complex number. Let us write the problem (1)-(4) as an operator boundary value problem. Let $H = L_2(0,\pi)$. Therefore, $H_1 = L_2(0,\infty;L_2(0,\pi))$. Define the domain set of Q(x) in H with the differential expression

$$-\frac{d^2}{dy^2} + q(x,y)$$

and the conditions (2)-(4) by

$$D(Q(x)) = (v(y): v'(y) \text{ absolutelycontinuos in } [0,\pi],$$
$$v(0) = v(\pi) = 0 \text{ and } v''(y) \in L_2(0,\pi)).$$

and its effect to v(y) by

$$Q(x)v = -\frac{d^2 v}{dy^2} + q(x,y)v.$$

Such defined operator Q(x) is a self-adjoint, bounded from bottom and inverse of it is a completely continuous operator for every $x \in [0,\infty)$. All conditions of Q(x) can be satisfied under additional conditions for q(x,y). (Albayrak, 1997).

Thus, if the boundary value problem (6)-(9) is written as an operator boundary value problem

$$\frac{d^4 u}{dx^4} + Q(x)u$$

$$u'(0) - h_1 u(0) = 0$$

$$u'''(0) - h_1 u''(0) = 0$$

in H_1, it is obtained that the result of the problem (6)-(9) has a Green function. The results obtained can be applied to the other examples in (Kostyuchenko and Levitan, 1967), (Dolph, 1961).

Green's Function of Differential Equations with Normal Operator Coefficient: In (Ozturk, 1998), denote the set of functions by $H_1 = L_2(a,b;H)$ $(-\infty \le a < x < b \le \infty)$ defined in the interval (a,b) and Bochner measurable.

Green function of operator formed by the differential expression

$$-y'' + Q(x)y, \qquad 0 \le x < \infty$$

in $L_2(0,\infty;H)$ space and the boundary condition

$$y'(0) - hy(0) = 0$$

has been studied, and it has been proved that Green function forms integral operator in the type of Hilbert-Schmidt. Here, Q(x) is **normal operator** whose inverse is compact at each value of x in $[0,\infty)$ and h is complex number.

Then, spectrum of operator formed by the differential expression

$$-y'' + Q(x)y, \qquad 0 \le x \le \pi$$

in $L_2(0,\pi;H)$ space and the boundary conditions

$$y'(0) - h_1 y(0) = 0$$

$$y'(\pi) + h_2 y(\pi) = 0,$$

has been studied and asymptotic expression of number of eigenvalues have been obtained. Here, Q(x) is self adjoint operator at each value of x in $[0,\pi]$. The results obtained have been applied to specific problems.

Example 1: Consider the following boundary value problem met in Quantum mechanics (Dolph, 1961).

$$-\Delta U + \iiint_D K(X,X')U(X')dX' = \lambda U \qquad (10)$$

$$U|_\gamma = 0 \qquad (11)$$

$$U'_X(0,X_2,X_3) - hU(0,X_2,X_3) = 0 \qquad (12)$$

where $X = (x_1, x_2, x_3)$, $X' = (x'_1, x'_2, x'_3)$ and D is a cylinderic region: $(x_2, x_3) \in \Omega$, $0 \le x_1 < \infty$, Ω is a finite region of plane, γ is a border of cylinder D, h is a complex number.

Let the kernel of integral operator in (10) satisfies the following conditions:

$$K(X,X') = K(X',X)$$

$$\iiint_D |K(X,X')|^2 dXdX' < \infty$$

and it can be shown as

$$K(X,X') = K_1(X - X') + K_2(X + X').$$

Operator $Q(x_1)$ mapping in space $L_2(\Omega)$ for every value of x in $[0,\infty)$ is defined with the formula

$$Q(x_1)U = -\frac{\partial^2 U}{\partial x_2^2} - \frac{\partial^2 U}{\partial x_3^2} + \iiint_D K(X,X')U(X')dX'.$$

Therefore, the boundary value problem (10)-(12) can be written as the problem

$$-\frac{d^2U}{dx_1^2} + Q(x_1)U = \lambda U$$

$$U'(0) - hU(0) = 0$$

Thus, special features of the Green function (10)-(12) can be examined.

Example 2: Consider the following boundary value problem

$$-\frac{\partial^2 U}{\partial x^2} + P(x)\left(-\frac{\partial^2 U}{\partial y^2} + y^2 U\right) = f(x,y) \qquad (13)$$

$$U_x(0,y) - hU(0,y) = 0 \qquad (14)$$

in region $0 \le x < \infty$, $-\infty < y < \infty$, where P(x) is a complex function satisfying the condition $|P(x)| \ge 1$, and h is a complex number. Let write the problem (13)-(14) as the boundary value problem with operator coefficient. Let define the Hermit operator, $B = -\frac{d^2}{dy^2} + y^2$ in space $H = L_2(-\infty,\infty)$. If the domain set of B is defined by

$D(B) = \{V(y) \in L_2(-\infty,\infty) : V'(y)$ is absolutely continuous in every finite subinterval of $(-\infty,\infty)$ and $-V'' + y^2V \in L_2(-\infty,\infty)\}$

and its effect to the element $V(y) \in D(B)$ is

$$BV = -V'' + y^2 V,$$

it is obvious that the B is the self-adjoint operator (Titchmarsh, 1962).

Let Q(x)=P(x)B. From $B = B^*$,

$$Q(x)^* = (P(x)B)^* = B^*\overline{P(x)} = \overline{P(x)}B$$

$$Q(x)Q^*(x) = P(x)B\overline{P(x)}B = \overline{P(x)}BP(x)B = Q^*(x)Q(x)$$

in every $x \in [0,\infty)$. That is, Q(x) is a normal operator in H. The problem (13)-(14) can be written as the equation with operator coefficient

$$-\frac{d^2U}{dx^2} + Q(x)U = f(x), \qquad f(x) \in L_2([0,\infty), H)$$

$$U'(0) - hU(0) = 0$$

in the space $H_1 = L_2([0,\infty), H)$. Thus the Green function of the problem (13)-(14) can be examined.

Example 3: Consider the boundary value problem

$$-\frac{\partial^2 U(x,y)}{\partial x^2} + P(x)\left(\frac{-\partial^2 U(x,y)}{\partial y^2} + y^2 U(x,y)\right) = \lambda U(x,y) \qquad (15)$$

$$U'_x(0,y) - h_1 U(0,y) = 0 \qquad (16)$$

$$U'_x(\pi,y) + h_2 U(\pi,y) = 0 \qquad (17)$$

in $0 \le x \le \pi$, $-\infty < y < \infty$, where P(x) is a continuous function satisfying the condition $P(x) \ge 1$ in $(-\infty,\infty)$, and $h_1 \ge 0$, $h_2 \ge 0$ are real numbers. Let $H = L_2(-\infty,\infty)$. Let write the problem (15)-(17) as the boundary value problem with operator coefficient in $H_1 = L_2(0,\pi;H)$.

$$LU = -\frac{d^2U}{dx^2} + Q(x)U, \qquad 0 \le x \le \pi \qquad (18)$$

$$U'(0) - h_1 U(0) = 0 \quad (19)$$

$$U'(\pi) - h_2 U(\pi) = 0 \quad (20)$$

where U(x) is an element of H in every $x \in [0, \pi]$, and Q(x) is an operator in H defined by

$$Q(x)U(x) = P(x)\left(-\frac{\partial^2}{\partial y^2} + y^2\right)U.$$

Let Q(x)=P(x)B, where B is an operator in the former example. From $B = B^*$, Q(x) will be a self-adjoint operator for every $x \in [0, \pi]$ in H. The spectrum of B includes only the eigenvalues $\beta_k = 2k + 1$, k=0,1,2,...(Titchmarsh, 1962). Therefore, it is obvious that the spectrum of the operator Q(x) for every x is only the eigenvalues

$\alpha_k(x) = P(x)\beta_k$, k=0,1,2, .

Hence, it is easily seen that Q(x) satisfies the following conditions.

1) The domain set of Q(x), D(Q(x))=D is independent from x and $\overline{D} = H$, where $\overline{D}$ shows the closure of D.
2) Assumed that the regular points set of Q(x) for every $x \in [0, \infty)$ belongs to $A = \{-\pi + \varepsilon_0 < \arg\lambda < \pi - \varepsilon_0 \ , \ 0 < \varepsilon_0 < \pi\}$.
3) $Q^{-1}(x)$ is a completely continuous operator in H for every x.

Thus, the resolvent of the problems (15)-(17) or (18)-(20) is a type of H-S (Hilbert-Schmidt) operator.

Let $H_1 = L_2(0, \infty; H)$. The all functions are defined in range $[0, \infty)$, their values belong to space H, and they are measurable in the meaning of Bochner in (Koklu, 1998).

In this study, in space H_1, it is investigated that Green's function (resolvent) of the operator formed by the differential expression

$$(-1)^n y^{(2n)} + Q(x)y, \qquad 0 \le x < \infty,$$

and boundary conditions

$$y^{(j)}(0) - h_j y^{(j-1)}(0) = 0, \qquad j = 1,3,\ldots\ldots,2n-1$$

where Q(x) is a **normal operator** that has pure discrete spectrum for every $x \in [0, \infty)$ in H. Assumed that domain set of Q(x) is independent from x and resolvent set of Q(x) belongs to $|\arg\lambda - \pi| < \varepsilon$ $(0 < \varepsilon < \pi)$ of complex plane λ. In addition, assumed that the operator function Q(x) satisfies the Titchmars-Levitan conditions. h_js are arbitrary complex numbers. The obtained result has been applied to examples.

Example 4: Consider the following boundary value problem in cylinderic region $\Omega = D \times [0, \infty)$ (D is a plane region)(n=2):

$$(-1)^n \frac{\partial^{2n} u}{\partial x_1^{2n}} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} + \int_\Omega K(x,s)u(s)ds = \lambda u \quad (21)$$

$$u\Big|_\Gamma = 0 \quad (22)$$

$$\frac{\partial^j u}{\partial x_1^j}(0) = 0 \qquad j=0,1,,\ldots,n-1 \quad (23)$$

where $x = (x_1, x_2, x_3)$, $s = (s_1, s_2, s_3)$, $u(x) = u(x_1, x_2, x_3)$, and Γ is a border of the region D. The kernel K(x,s) is a function such that

$$K(x,s) = \alpha[f(x+s) + g(x-s)] + \beta\left[\overline{f(x+s)} + \overline{g(x-s)}\right]$$

that is satisfies the conditions

$$\int_\Omega \int_\Omega |K(x,s)|^2 dxds < \infty$$

$$\left.\frac{\partial^j K(x,s)}{\partial x_1^{\,j}}\right|_{x_1=0} = 0, \qquad j=0,1,\ldots,n-1.$$

g(x-s) is a function satisfying the condition g(x-s)=g(s-x) and α, β are any complex numbers ($|\alpha| = |\beta| = 1$). Therefore, $\alpha A + \beta A^*$ is a normal operator.

Let $H = L_2(D)$ and n=2. Let M is a self-adjoint operator formed by the differential expression $-\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$ in H and the boundary condition (22), and

$$Au = \int_\Omega K(x,s)u(s)ds$$

is a bounded operator for every $x_1 \in [0, \infty)$. Integro-differential expression (21) can be written as the differential expression with operator coefficient

$$(-1)^n \frac{d^{2n} u}{dx_1^{2n}} + Q(x_1)u \quad (24)$$

where

$$Q(x_1) = -\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \int_\Omega K(x_1, x_2, x_3; s)u(s)ds$$

in space $L_2(0, \infty; H)$.

Investigating of the Green's function of the problem (21), (22), (23) can be reduced examining the boundary value problem (23), (24). Considering that the operators A and M can exchange their places, it is seen that operator $Q(x_1)$ is a normal operator for every $x_1 \in [0, \infty)$. Since the inverse of M is compact and operator A is bounded, it is obvious that spectrum of operator $Q(x_1)$ is discrete. It can be shown that $Q(x_1)$ also satisfies the other conditions of Q(x) in (Koklu, 1998). Similar example for a second order differential operator has been given in (Kleiman, 1977).

Example 5: As a special case, if space H is n(finite)-dimensional, the problem is written such that

$$y_j^{IV} + \sum_{K=1}^{n} q_{jk}(x) y_k \text{ ,j=1,2,...,n} \qquad (25)$$

$$y_j^{(K)}(0) - h_n y_j^{(K-i)}(0) = 0 \qquad \text{K=1,3 (26)}$$

where $Q(x) = (q_{jk})$ is an $n \times n$ matrix function.

The condition $Q(x)Q^*(x) = Q^*(x)Q(x)$ can be satisfied by adding some other conditions on complex valued functions $\{q_{jk}(x)\}$. Therefore, existing of Green function of operator formed with the expression (25) and the conditions (26) is obtained.

Example 6: Proof of existence of the operator valued function Q(x) satisfied conditions 1-) - 6-) using (Kleiman, 1977).

Let $H = l_2$, $Q(x) = m\begin{pmatrix} q_1(x) & 0 & 0 & \dots & 0 & \dots \\ 0 & q_2(x) & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_j(x) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$

where m is an arbitrary complex number in S_ε, $q_j(x) = K_j x + d$, d=const>0, K_j are any positive numbers satisfying $\sum_{j=1}^{\infty} \frac{1}{K_j} < \infty$ while $K_1 \le K_2 \le \dots \le K_n \le \dots$. It was shown that Q(x)'s all conditions were provided.

Let's pay attention that it is enough to indicate Q(x) satisfies the conditions almost everywhere.

1. Q(x) is a normal operator for every x>0 in l_2 because of $Q^*(x) = \overline{m} Q(x)$ ($\overline{m}$ is a complex conjugate of m).

2. $Q^{-1}(x) = m^{-1}\begin{pmatrix} q_1^{-1}(x) & 0 & 0 & \dots & 0 & \dots \\ 0 & q_2^{-1}(x) & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q_n^{-1}(x) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$

$$q_j^{-1}(x) = (K_j x + d)^{-1} \underset{j \to 0}{\to} 0, \quad (x \neq 0).$$

This also demonstrates that $Q^{-1}(x)$ is a compact operator in every x $(x \neq 0)$.

3.

$$(Q(x) - \lambda I)^{-1} = \begin{pmatrix} (mq_1(x)-\lambda)^{-1} & 0 & 0 & \dots & 0 & \dots \\ 0 & (mq_2(x)-\lambda)^{-1} & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (mq_n(x)-\lambda)^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This statement proves that $(Q(x) - \lambda)^{-1}$ is exist and bounded at every point in $\lambda \in (-\infty, 0)$. That is, the resolvent set of Q(x) is in S_ε.

4.

$$\int_0^\infty \sum_{j=1}^{\infty} \frac{1}{q_j^{7/4}(x)} dx = \int_0^\infty \left(\sum_{j=1}^{\infty} \frac{1}{(K_j x + d)^{7/4}} \right) dx$$

$$= \sum_{j=1}^{\infty} \frac{1}{K_j} \int_0^\infty \frac{d(K_j x)}{(K_j x + d)^{7/4}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{K_j} \int_0^\infty \frac{dt}{(t+d)^{7/4}} = \sum_{j=1}^{\infty} \frac{1}{K_j} \frac{4}{3} = \frac{4}{3} \sum_{j=1}^{\infty} \frac{1}{K_j} < \infty$$

5. $q_j^{-1/4}(x) q_j^{1/4}(\xi) = (K_j x + d)^{-1/4} (K_j \xi + d)^{1/4}$

$$(K_j x + d)^{-1/4} (K_j \xi + d)^{1/4} \le (K_j x + d)^{-1/4} (K_j x + x + d)^{1/4}$$

$$= \left(1 + \frac{d}{K_j x + d}\right)^{1/4} \le \left(1 + \frac{d}{d}\right)^{1/4} = \sqrt[4]{2}$$

while $|x - \xi| \le 1 \Leftrightarrow -1 \le \xi - x \le 1 \Rightarrow \xi \le x + 1$.

6. $|x - \xi| \le 1$; It was taken that a=1 satisfying condition $0 < a < \frac{5}{4}$. Assumed that $x \ge 1$ as in general.

$$\left|q_j^{-1}(x)[q_j(\xi) - q_j(x)]\right| = (K_j x + d)^{-1} |K_j \xi - K_j x|$$

$$= |x - \xi| K_j (K_j x + d)^{-1} = |x - \xi| \left(x + \frac{d}{K_j}\right)^{-1}$$

$$\le |x - \xi| \left(1 + \frac{d}{K_j}\right)^{-1} \le |x - \xi|.$$

Thus $\left|q_j^{-1}(x)[q_j(\xi) - q_j(x)]\right| \le |x - \xi|$ while $|x - \xi| \le 1$.

Therefore, it has shown that conditions 1-) - 6-) were satisfied.

Example 7: Let $\Omega \subset R^m \quad (m \geq 1)$ be any finite region with uniformly smooth boundary and $R^+ = [0,\infty)$.

Let us consider the boundary value problem of

$$\frac{\partial^4 u}{\partial x^4} + q(x)(-\Delta_y)^s u = \lambda u \qquad (27)$$

$$\left.\frac{\partial u(x,y)}{\partial x}\right|_{x=0} + h_1 u(0,y) = 0 \qquad (28)$$

$$\left.\frac{\partial^3 u(x,y)}{\partial x^3}\right|_{x=0} + h_2 u(0,y) = 0 \qquad (29)$$

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \gamma}\right|_{\partial\Omega} = ... = \left.\frac{\partial^{s-1}}{\partial \gamma^{s-1}}\right|_{\partial\Omega} = 0 \qquad (30)$$

in space $L_2(R^+ \times \Omega)$. Here $y=(y_1,y_2,....,y_m)$,

$$-\Delta_y = -\frac{\partial^2}{\partial y_1{}^2} - \frac{\partial^2}{\partial y_2{}^2} - ...... - \frac{\partial^2}{\partial y_m{}^2}$$

s is any integer $> \frac{2m}{7}$, $\partial\Omega$ is the boundary of Ω region, γ is the normal of $\partial\Omega$ and q(x) is a complex valued function with values in $C \backslash S_\varepsilon$ satisfying the conditions

$$c_1(1+x^\alpha) \leq |q(x)| \leq c_2(1+x^\alpha)$$

where $\alpha > \frac{4}{7}$, c_1, c_2 are positive constants and h_1, h_2 are arbitrary complex constants.

Let us define self-adjoint A operator (like in (Otelbayev, 1990) in space $H = L_2(\Omega)$ by $(-\Delta_y)^s$ with boundary conditions (30).

Therefore the problem (27)-(30) in the space $H_1 = L_2(R^+ \times \Omega) = L_2(R^+, H)$ can be written as a boundary value problem with operator coefficient as follows:

$$\frac{d^4 u}{dx^4} + Q(x)u - \lambda u = 0$$

$$u'(0) - h_1 u(0) = 0$$

$$u'''(0) - h_2 u'(0) = 0$$

where u(x)=u(x, .), Q(x)=q(x)A.

Resolvent set of operator function Q(x) defined like this consists of region S_ε and it can be shown that conditions 1-) - 6-) are satisfied (See also (Kostyuchenko and Levitan, 1967), (Boymatov, 1973). Applying the founded results in the theoretical part, Green function of the problem (27)-(30) can be examined.

Results and Discussion

This paper described Green's Function and some applications of various differential equations with different operator coefficient. One could say that solutions of the problems on spectral analysis of self-adjoint ordinary differential equations, such as investigating of spectrum, expansion to eigen functions, asymptotic behaviour of discrete spectrum, etc., have completed. Although there have been many researches on the spectral analysis of differential expressions with operator coefficient, there are still many problems which needs to be studied for this kind of operators.

Using the parametrics method, resolvent of the operator formed by boundary conditions (2) and the differential expression

$$(-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)}$$

where $Q_j(x)$'s (j=1,2,....,2n) are the operator valued functions, can be investigated.

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