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Estimation of Generalized Logistic Distribution by Probability Weighted Moments

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Abstract: The parameters of Generalized Logestic Distribution are estimated by method of Probability Weighted Moments (PWM). Finite sample properties of PWM estimators are investigated through computer simulations and PWM estimators seem to outperform the Method Of Moments (MOM) particularly for small sample in terms of bias, variances and Root Mean Squre (RMS).

Key words: Logistic Distribution, Probability Weighted Moments

Introduction

The Generalization of the logistic distribution differs from others which have been defined in the literature (e.g Gumble, 1944; Dubey 1969) it has not previously been used in statistics. The name is chosen to reflect the distribution's similarity to the Generalized Pareto and generalized extreme value distribution it can be used for modeling extremes of natural phenomena, and is of considerable importance in hydrology. Currently favored methods of estimation of the parameters and quantiles of the distribution are Jenkinson's (1969) method of settles and the method of maximum likelihood Prescott and Walden, 1980, 1983). Neither method is completely satisfactory. The justification of the maximum likelihood approach is based on large sample theory, and there has been little assessment of the performance of the method when applied to small or moderate samples whereas the sextile method involves an inherent arbitrariness (why sextiles rather than say quartiles or octiles?), requires interpolation in a table of values of a function in order to estimate the shape parameter of the distribution and has statical properties that are not known even for large samples.

Probability weighted moments, a generalization of the usual moments of a probability distribution was introduced by Green *et al* (1979). There are several distribution e.g. the Gumbel, Logistic and Weibull whose parameters can be conveniently estimated from their probability weighted moments. The Gumbel distribution, is one of the special case of the generalized extreme value distribution. Landwehr *et al.* (1979) investigate the small sample properties of Probability Weighted Moment (PWM) estimators of parameters and quantiles for the Gumbel distribution and found them superior on many respects to the conventional moment and maximum-likelihood estimators.

In this article we summarize some theory for probability weighted moments and show that they can be used to obtain estimates of parameters.

Materials and Methods

Probability Weighted Moments

Definition: Let x be a real valued random variable with distribution function F Greenwood *et al.* (1979)defined the probability weighted moments of x to be the quantities.

$$M_{p,r,s} = E\,[X^p\{F(x)\}^r\{1-F(X)\}^s]$$

$$= \int X^p \{F(X)\}^r \{1-F(X)\}^s \, dF(x)$$

Where p,r and s are real numbers Note that $M_{p,r,s}$ exists for all r, s $\geq \theta$ if and only if $E|X|^p$ exists.

We are usually interested in cases in which r and s are positive integers we then have

$$M_{p,r,s} = \frac{r!s!}{(r+s+1)!} \quad EX^p_{r+1:\,r+s+1}$$

Where $X_{k:n}$ is the kth order statistic of a random sample of size n from the distribution F.

The definition of $M_{p,r,s}$ is valid both for continuous and for discrete random variables. In the former case we have

$$M_{p,r,s} = \int x^p \{F(x)\}^r\{1-F(x)\}^s f(x)\,dx$$

Where f is the probability density function of X. We also have

$$M_{p,r,s} = \int_0^1 \{X(F)\}^p F^r (1-F)^s dF$$

When x(F) is the quantile function of X.

The quantities $M_{p,r,s}$ may be used to describe the characteristics of probability distribution. One possible approach is to work with $M_{p,o,o}$, p=1, 2, ...,these are just the conventional non central moments of X. We shall instead work with the moments $M_{1,r,s}$, in to which X enters linearly and in particular with the quantities (which we shall also refer to as PWM_s)

$\alpha_r = M_{1,s,r,} = E[X\{1-F(x)\}^r]$ r=θ ,1,.....

$\beta_r = M_{1,r,s,} = E[X\{F(x)\}^r]$ r=θ,1,....

Note that $r\alpha_{r-1} = EX_{1;r}$ and $r\beta_{r-1} = EX_{r:r}$ are expected values of extreme order statistics.

The characterization of a distribution by the α_r and β_r are functions of each other i.e.

$$a_r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \beta_k$$

and

$$\beta_r = \sum_{k=o}^{r} (-1)^k \binom{r}{k} \alpha_k$$

Given a random sample of size n from the distribution F, estimation of β_r is most conveniently based on the ordered sample $x_1 \leq x_2 \leq \ldots x_n$ The statistic

$$b_r = n^{-1}\sum_{i=1}^{n} \frac{(i-1)(i-2)\ldots(i-r)X_i}{(n-1)(n-2)\ldots(n-r)}$$

is an unbiased estimator of (Landwehr *et al.*, 1979)

L-Moments: Probability weighted moments characterize a distribution but are not particularly meaningful on themselves, it is therefore, useful to define function of PWM, which give a descriptive summary of the location, scale and shape of a probability distribution.

We define the L-moments of a real valued random variable X to be the quantities

$$\lambda_r = r^{-1}\sum_{k=o}^{r-1} (-1)^k \binom{r-1}{k} EX_{r-k:r} \quad r = 1,2,\ldots$$

The L in "L-moments" emphasizes that λ_r is a linear function of the expected order statistics. An L-statistic in terms of the previously defined α_r and β_r we have

$$\lambda_{r+1} = (-1)^r \sum_{k=o}^{r} P_{r,k}\alpha_k = \sum_{k=o}^{r} P_{r,k}\beta_k \quad r = 0,1,\ldots$$

Where

$p_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r-k}{k}$

PWM estimators for the generalized logistic distribution.
The probability distribution function is

$$f(x) = \frac{\alpha^{-1} e^{-(i-k)y}}{(1+e^{-y})^2}$$

Where

$Y = -K^{-1} \log \{1-k(x-\xi)/\alpha\} \quad k \neq \theta$

$\quad = (x-\xi)/\alpha \quad k = \theta$

$\xi + \alpha/k \leq x < \infty \quad$ if $k < \theta$

$-\infty < x < \infty \quad$ if $k = \theta$

$-\infty < x \leq \xi + \alpha/k \quad$ if $k > \theta$

$$F(x) = \frac{1}{1+e^{-y}}$$

$x(F) = \xi + \alpha[1-\{(1-F)/F\}^k/k \quad k \neq 0$

$\quad = \xi - \alpha \log\{(1-F)/F\} \quad k = 0$

When k=θ it is the logistic distribution. The probability weighted moments of the GL distribution for k≠θ are given by

$$r\alpha_{r-1} = \xi + \frac{\alpha}{k}\left\{1 - \frac{\Gamma(1-k)\Gamma(r+k)}{r(r)}\right\} \quad |k|<1$$

$$r\beta_{r-1} = \xi + \frac{\alpha}{k}\left\{1 - \frac{\Gamma(1+k)\Gamma(r-k)}{\Gamma(r)}\right\} \quad |k|<1$$

and the L-moments are

$$\lambda_1 = \xi + a\{1 - \Gamma(1-k)\Gamma(1+k)\}/k$$

$$\lambda_2 = a\Gamma(1-k)\Gamma(1+k)$$

$$\lambda_3 = -ka\Gamma(1-k)\Gamma(1+k)$$

The estimates of parameters are calculated by the above three L-moments and found to be

$$k = -\lambda_3/\lambda_2$$

$$\propto = \lambda_2/\{\Gamma(1-k)\Gamma(1+k)\}$$

$$\xi = \lambda_1 - a\{1-\Gamma(1-k)\Gamma(1+k)\}/k.$$

(See Appendix)

Results and Discussion

Simulation results: Computer simulation (1000) are performed for different sample sizes (20, 50) and for different parameters (ξ = 0, α =1, k=-0.2 and ξ =0, α= 1, k=0.2) to estimate the parameters of Generalized logistic distribution by Method of moments and probability Weighted Moments method. The bias and RMS are calculated. The PMW estimators seem to out perform MOM in both bias and RMS.

Appendix: For the GL distribution, estimates of the parameters by the method of Probability Weighted Moments can be calculated as:

$\beta_r = M_{1,r,\varnothing}$

$$= \int_0^1 [\xi + a[1-\{(1-F)/F\}^k]/k]F^r\, dF$$

substituting $\mu = \frac{1-F}{F}$

$$\int_0^\infty \frac{\xi}{(1+u)^{r+2}}du + \int_0^\infty \frac{\propto}{k}\cdot\frac{1}{(1+u)^{r+2}}du - \int_0^\infty \frac{\propto}{k}\cdot\frac{\mu^k}{(1+u)^{r+2}}du$$

Applying the result $\beta_{(m,n)} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{X^{m-1}}{(1+x)^{m+n}}dx$

$$r\beta_{r-1} = \xi + \frac{a}{k}\left\{1 - \frac{\Gamma(1+k)\Gamma(r-K)}{\Gamma(r)}\right\}$$

From here L-moments and then the estimates of the parameters can be calculated.

Comparison of PWM method and MOM

n	Parameters	PWM Bias	PWM RMS	MOM Bias	MOM RMS
20	ξ=0	-0.2450	0.306140	-0.3667	0.33724
	α=1	0.0895	0.045144	0.1421	0.10505
	k=0.2	0.0224	0.012669	-0.0952	0.02255
	ξ=0	-0.1044	0.188225	0.1423	1.02721
	α=1	0.0115	0.040855	0.0913	0.12662
	k=-0.2	0.0448	0.020010	0.2915	0.08385
50	ξ=0	-0.0562	0.068456	-0.1945	0.10949
	α=1	0.0281	0.181870	0.0798	0.3551
	k=0.2	0.0613	0.011400	-0.0778	-0.0993
	ξ=0	-0.0427	0.063776	0.3732	0.47464
	α=1	0.0039	0.016760	0.0351	0.08089
	k=-0.2	0.0176	0.009589	0.2846	0.08483

Conclusion

The estimation of the parameters by the PWM method is quite simple and easy. The simulation results show that for the generalized logistic distribution, the PWM are less biased and having smaller RMS as compared to MOM. Also, as the sample size increases the bias and RMS decreases.

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