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Conservation of "Partial Energy", with an Application on Water Wave Dissipation

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Abstract: An extension of the energy conservation theorem is established for fluid dynamical purposes. Equations are developed that divide the mechanical energy of a system into three "partial energies" based on the three orthogonal directions of a Cartesian coordinate system. In addition to terms similar to the terms of ordinary energy equations, these equations contain terms that handle transfer of energy within a fluid, i.e. between the three partial energies. Thereby internal transfer of energy can be studied. A brief study, that gives a consequence of viscous dissipation on water waves, is included as an example.

Key words: Partial Energy, Energy, Fluid Dynamics, Water Waves

Introduction

The kinetic energy of a laminar, unidirectional flow, is due to fluid velocities in one single direction. When turbulence or waves are generated, e.g. by the Kelvin-Helmholtz instability, energy is transferred internally from the unidirectional flow to multidirectional flows. It implies that the kinetic energy, that was initially due to fluid velocity in a single direction, is now also due to velocity components at right angles to the original direction. When calculating ordinary kinetic and potential energy (mechanical energy), directional information is removed. Here, the aim is to establish the additional equations needed to make this information available. Thereby we learn what is needed to pass energy from velocities of one direction to another by internal forces.

Lagrangian Conservation Equation: Conservation equations for three "partial energies" E_1, E_2 and E_3, based on the three orthogonal directions x_1, x_2 and x_3 of a Cartesian coordinate system, are established. They treat the mechanical energy of a fluid with density ρ, inside a closed material surface A surrounding a volume V. For the sake of simplicity the x_3 axis is defined vertical, so that potential energy is assumed to be due to the acceleration of gravity g acting in the negative x_3 direction. Then, the i^{th} partial energy is defined as

$$E_i = \int_V \tfrac{1}{2}\dot{x}_i^2 \rho dV + \delta_{i3}\int_V x_3 g\rho dV \ , \qquad (1)$$

where i can take the values 1, 2 and 3. The Kronecker delta $\delta_{ij} = 1$ when $i = j$, else $\delta_{ij} = 0$. Further, the dot above x is a total time derivative, which is also written D/Dt. No sum over equal indices is assumed.

According to (1), the partial energy consists of two terms, a kinetic energy term and a potential energy term. It is easily verified that the total mechanical energy $E = E_1 + E_2 + E_3$. Partial conservation equations are developed by treating each component of Newton's 2nd law separately. Since Newton's 2nd law treats a specific mass, Lagrangian conservation equations are obtained. They are finally transformed to Eulerian form for an incompressible fluid.

Initially, Newton's 2nd law is adopted on an infinitesimal mass dm with density ρ and volume dV inside a closed surface dA. The external forces are split into gravity forces and surface stresses. Then the i^{th} component of Newton's 2nd law reads

$$-\delta_{i3} g\rho\, dV + dF_i = \ddot{x}_i \rho\, dV \ . \qquad (2)$$

The first term of (2) is the gravity, while dF_i is the component of the stress forces that acts in the direction of the x_i axis on the surface of dV.

Now (2) is multiplied by $\dot{x}_i$:

$$-\delta_{i3}\dot{x}_i g\rho\, dV + \dot{x}_i dF_i = \dot{x}_i \ddot{x}_i \rho\, dV \ . \qquad (3)$$

By definition, the mass of a material volume $dm=\rho\, dV$, is constant. Further $x_i\, \delta_{i3} = x_3\, \delta_{i3}$, and therefore the left term of (3) can be transformed as follows:

$$\delta_{i3}\dot{x}_i g\rho\, dV = \frac{D}{Dt}\left(\delta_{i3} x_3 g\rho\, dV\right) \qquad (4)$$

Due to the Kronecker delta, this term only applies to the third direction. It is the rate of change of the potential energy of dm.

Since $\rho\, dV$ is constant, the term on the right hand side of (3) can be transformed in a similar way:

$$\dot{x}_i \ddot{x}_i \rho\, dV = \frac{D}{Dt}\left(\tfrac{1}{2}\dot{x}_i^2 \rho dV\right). \qquad (5)$$

This is the rate of change of a term that may be denoted "the partial kinetic energy of dm". (If the linear dimension of dV is denoted l, its translational partial kinetic energy is proportional to l^3 while its rotational partial energy is proportional to l^5. Hence - as for (ordinary) kinetic energy - rotational energy does not contribute on an infinitesimal scale.)

The sum of the terms in the brackets of (4) and (5), is the partial energy of dm. By denoting the sum dE_i, (3) gets the form

$$\frac{D}{Dt}\left(dE_i\right) = \dot{x}_i dF_i \ . \qquad (6)$$

The stress force component acting on a small part da of the surface dA of dV, is

$$df_i = \mathbf{i}_i \cdot \sigma \cdot \mathbf{n}\, da, \qquad (7)$$

where σ is the stress tensor, $\mathbf{i}_i$ the unit vector in the direction of the x_i axis, and $\mathbf{n}$ a unit vector normal to da that is pointing out of dV. By integrating over dA, Gauss' theorem implies that the i^{th} component of the surface forces on dV

$$dF_i = \mathbf{i}_i \cdot (\nabla \cdot \sigma) dV\,. \qquad (8)$$

The integration sign is omitted in (8), since dV is infinitesimal so that the integrand can be regarded as a constant.

By neglecting bulk viscosity, the stress tensor reads:

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\sum_{k=1}^{3}\frac{\partial \dot{x}_k}{\partial x_k}, \qquad (9)$$

according to Landau and Lifshitz (1966) eqn. (15.3). Here p is the pressure, and μ the dynamic viscosity.

When (8) is inserted into (6) and the equation obtained is integrated over V, then

$$\dot{E}_i = \int_V \dot{x}_i\, \mathbf{i}_i \cdot (\nabla \cdot \sigma) dV$$

$$= \int_V \nabla \cdot (\sigma \cdot \mathbf{i}_i \dot{x}_i) dV - \int_V (\mathbf{i}_i \cdot \sigma) \cdot \nabla \dot{x}_i\, dV \qquad (10)$$

The first term on the second line side is denoted W_i:

$$W_i = \int_V \nabla \cdot (\sigma \cdot \mathbf{i}_i \dot{x}_i) dV\,. \qquad (11)$$

By adopting Gauss' theorem

$$W_i = \int_A \mathbf{n} \cdot \sigma \cdot \mathbf{i}_i \dot{x}_i\, dA\,. \qquad (12)$$

Here $\mathbf{n} \cdot \sigma \cdot \mathbf{i}_i$ is the component of the surface stresses in the direction of $\mathbf{i}_i$. Since it is multiplied by $\dot{x}_i$, which is the velocity component in the same direction, W_i is the work performed by the external force component in the direction of $\mathbf{i}_i$ on the surface of V.

The last term of (10) is denoted Q_i. Hence

$$Q_i = -\int_V (\mathbf{i}_i \cdot \sigma) \cdot \nabla \dot{x}_i\, dV \equiv -\int_V \sum_{j=1}^{3} \sigma_{ij} \frac{\partial \dot{x}_i}{\partial x_j} dV\,. \qquad (13)$$

By defining

$$T_i = \int_V p \frac{\partial \dot{x}_j}{\partial x_i} dV \qquad (14)$$

and

$$D_i = \mu \int_V \sum_{j=1}^{3} \left[\left(\frac{\partial \dot{x}_i}{\partial x_j} \right)^2 + \frac{\partial \dot{x}_j}{\partial x_i}\frac{\partial \dot{x}_i}{\partial x_j} - \frac{2}{3}\delta_{ij}\frac{\partial \dot{x}_i}{\partial x_i}\sum_{k=1}^{3}\frac{\partial \dot{x}_k}{\partial x_k} \right] dV \qquad (15)$$

then

$$Q_i = T_i - D_i \qquad (16)$$

is obtained by inserting σ_i from (9) into (13).

By summing $D_1 + D_2 + D_3$, the expression for the total dissipation of the system is achieved. According to Liggett (1994), p. 28, the argument of the sum over j can be written on quadratic form. Hence, not only the sum of D_i is positive, but each of the three D_i are positive as well, and determine the actual part of the total dissipation.

The definitions of W_i, D_i and T_i imply that (10) gets its final Lagrangian form:

$$\dot{E}_i = W_i - D_i + T_i\,. \qquad (17)$$

This is the conservation equation for partial energy of a material system, which is similar to a conventional conservation equation for mechanical energy. It says that the rate of change of the partial energy is due to the external work W_i performed by the actual external force component, due to a dissipation term D_i, and a term T_i.

If the three partial conservation equations are added, the ordinary energy equation of the system is achieved. If the sum of T_i is negative, it is the amount of energy that is removed from the three E_i to compress the fluid. If it is positive, it is the amount of energy that is received by them because of decompression. But separately, the three T_i have a new and additional function as transfer terms. They transfer energy inside V, between the three partial energies. This is readily seen, by taking the sum of them:

$$T_1 + T_2 + T_3 = \int_V p \nabla \cdot \mathbf{v}\, dV\,, \qquad (18)$$

where $\mathbf{v}$ is the velocity vector. According to the equation of continuity, the integrand of (18) vanishes everywhere for an incompressible fluid. So, if E_1 obtains energy because $T_1 > 0$, at least one of the two other transfer terms is negative in order to remove energy from E_2 or E_3 in such a way that the total energy is conserved. Hence the T_i terms work as transfer terms of energy between the partial energies, and the value of a T_i determines the amount of energy that is transferred to E_i from the two other partial energies of the system. Thereby information is obtained regarding the instability of the system.

Eulerian Conservation Equation for Incompressible Fluids: In order to get (17) on Eulerian form, the left hand side of it is split into a partial time differentiation term and a convective term. By defining

$$e_i = \rho(\tfrac{1}{2}u_i^2 + g x_3 \delta_{i3}) \qquad (19)$$

the following expression is valid for the fluid inside the volume V surrounded by a fixed surface A:

$$\dot{E}_i = \int_V \frac{de_i}{dt} dV \qquad (20)$$

For an incompressible fluid

$$\int_V \frac{de_i}{dt} dV = \int_V \frac{\partial e_i}{\partial t} dV + \int_V \mathbf{v} \cdot \nabla e_i dV \qquad (21)$$

where v is the velocity vector. Since $\nabla \cdot \mathbf{v} = 0$, (20) and (21) imply

$$\dot{E}_i = \int_V \frac{\partial e_i}{\partial t} dV - \int_A \mathbf{n} \cdot \mathbf{v} e_i dA . \qquad (22)$$

Above, the components of **v** are denoted u_i. They replace $\dot{x}_i$ in the following.

On the right hand side of (22), the second term is denoted S_i. It is the flow of partial energy to the system through A. Both terms are similar to the terms of the ordinary energy equation. They do therefore not add any new information, apart from giving the information on component form. The same is the case for the terms that depend on compression. The simpler incompressible form is therefore chosen.

The equation of partial energy on Eulerian form for an incompressible fluid is obtained by exchanging the left hand side of (17) with the expression from (22):

$$\frac{\partial E_i}{\partial t} = S_i + W_i - D_i + T_i . \qquad (23)$$

A consequence on Wave Dissipation: In order to show one way to use the partial energy equations, an application is shown in the following that treats wave dissipation. The intention is not to give a full solution, merely to demonstrate a simple way to achieve some information that is not easily obtained by other means.

The waves are described as regular, linear, deep water waves, with the x_1 axis in the direction of the wave propagation and the x_3 axis upwards. The wave height H is a weak function of the time in order to allow dissipation.

An Eulerian approach is adopted. The system is the water inside an immaterial rectangular box with sides parallel to the axes. The length of the box is a whole number of wave lengths, its bottom is at the bottom of the sea, and its top above the water. For this system, the net flows of partial energy through the boundaries cancel. Hence, according to (23), any time dependency of an E_i is due to D_i, W_i and T_i.

According to Svendsen and Jonsson (1976), the fluid motion of deep water regular waves is described by circular paths where the speed of the fluid is constant. Hence the kinetic energy is evenly distributed between E_1 and E_3, while the potential energy contributes to E_3 only. According to Phillips (1977) p. 38, "the mean kinetic and potential energies are equal" for "any conservative dynamical system undergoing small oscillations". As the viscous damping is assumed to be small, and an exact relationship is not needed here, the relationship given by Phillips is sufficiently good for our needs. Hence

$$E_3 = 3E_1, \qquad (24)$$

where E_i is now partial energy *per unit surface area*. According to Svendsen and Jonsson (1976), p. 109, the total mechanical energy of Stokes waves per unit surface area, is given as

$$E = \tfrac{1}{8} \rho g H^2 . \qquad (25)$$

Since $E = E_1 + E_3$,

$$E_1 = \tfrac{1}{32} \rho g H^2 \qquad (26)$$

and

$$E_3 = \tfrac{3}{32} \rho g H^2 . \qquad (27)$$

Similarly, the total dissipation per unit surface area $D = D_1 + D_3$. By inserting for u_1 and u_3 in (15) and exchanging μ/ρ by the kinematic viscosity ν, it is seen that

$$D_1 = D_3 = 2\nu k^2 E . \qquad (28)$$

Since $D_1 = D_3$, both partial energies loose the same amount of energy by dissipation, while the relationship between the partial energies requires that E_3 looses three times more energy than E_1. So transfer of energy between E_1 and E_3 is required, and therefore non-zero transfer terms are needed.

The mean values of the transfer terms per unit surface area, T_1 and T_3, are found by using that

$$T_1 = -T_3, \qquad (29)$$

and applying this in (17). Both W_1 and W_3 vanish. Hence for $i = 1$ (17) reads

$$\dot{E}_1 = -D_1 + T_1 , \qquad (30)$$

and for $i = 3$, when exchanging D_3, E_3 and T_3 by D_1, E_1 and T_1 using (25), (29) and (30), it reads

$$3\dot{E}_1 = -D_1 - T_1 . \qquad (31)$$

Therefore the transfer term per unit surface area

$$T_1 = \tfrac{1}{2} D_1 = \nu k^2 E . \qquad (32)$$

In order to see to what extent Stokes' (1847) wave theory allows transfer, the transfer term of the waves is calculated. Regular waves with circular frequency ω and wave number k are described by a velocity potential of the form

$$\phi = -\sum_i a_i(x_3) \sin i(\omega t - k x_1) , \qquad (33)$$

where $a_i(x_3)$ is the amplitude of term i. According to potential theory, the dynamic pressure

$$p = -\rho \frac{\partial \phi}{\partial t} = \rho \sum_i i a_i(x_3) \omega \cos i(\omega t - k x_1) . \qquad (34)$$

Further

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial^2 \phi}{\partial x_1^2} = \sum_i i^2 a_i(x_3) k^2 \sin i(\omega t - k x_1) . \qquad (35)$$

Hence the mean value of the integrand of the transfer term given in (14), $p \partial u_1 / \partial x_1$, vanishes everywhere. So – since Stokes waves are stable – they are stable also in the sense that no energy is transferred between E_1

and E_3. This result is not restricted to Stokes waves. Since the amplitudes a_i are not specified, the result is valid for all regular waves that can be described by (33), regardless of the amplitudes a_i. The sum vanishes for a series of superimposed waves too. Hence transfer has to be obtained by other means.

We may ask how the flow that makes the transfer possible, is obtained. As treated here, this theory does not tell. It merely tells to what extent the flow supports the transfer. As the potential energy is the source to the problem, and the potential energy of waves is obtained above the trough level, we must expect to find the additional velocity terms mainly above the trough level. Whether they appear as waves or turbulence is beyond the scope of this paper.

Conclusion

A new set of energy conservation equations is established. A term is established that takes care of the transfer of energy between the three orthogonal directions in a Cartesian coordinate system. The example in the previous section shows one way to use them. They can probably be used to solve other problems too, and by better methods than shown here. In particular they appear useful for studying waves and instable flows, as they separate the energy of currents from the energy of waves and turbulence, and provide means to calculate the transfer of energy between them.

A similar set of equations is developed for angular momentum. They are published in a separate paper – Naeser (2002) – where other consequences of wave dissipation are shown.

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