Pakistan Journal of Applied Sciences 2(6): 682-685, 2002

Conservation of "Partial Angular" Momentum with Applications on Water Waves

Harald Naeser
Agder University College, Faculty of Technology and Science
Groosev. 36, N – 4876 Grimstad, Norway

Abstract: An extension of the conservation equation for angular momentum is established for fluid dynamical purposes. The three components of the angular momentum vector, described in a Cartesian coordinate system, are divided into pairs of "partial angular momentum" terms. Conservation equations are developed for each of these terms. They show that shear stresses (viscous and Reynolds stresses) transfer angular momentum between the two terms of a pair, and they provide means to calculate internal transfer of angular momentum between different dynamical regimes of a system. Two such regimes are waves and shear currents, and a brief study of the influence of shear stresses on waves and waves' interactions with currents, is included as an example.

Key words: Partial Angular Momentum, Angular Momentum, Fluid Dynamics, Water Waves

Introduction

In fluid dynamics, angular momentum is hardly in use at all. The aim of this paper is to make the angular momentum concept more useful for fluid dynamical purposes. This is achieved by dividing each component of the angular momentum vector into two partial angular momenta, and establish conservation theorems for each of them in a similar way as for partial energy, which was established in Naeser (2002).

The angular momentum of a laminar, unidirectional flow is due to fluid velocities in one single direction. When turbulence or waves are generated, e.g. by the Kelvin-Helmholtz instability, angular momentum (as well as energy) is transferred by internal forces from the unidirectional flow to multidirectional flows by a process that conserves angular momentum. It implies that an angular momentum vector, initially due to fluid velocities in a single direction, will be due to velocity components at right angles to the original direction too. Here, the aim is to establish the additional equations needed to make the transfer information available. Thereby we learn what is needed to transfer angular momentum from velocities of one direction to another by internal forces.

Lagrangian Conservation Equation: Conservation equations for six partial angular momenta P_{12}, P_{13}, P_{21}, P_{23}, P_{31} and P_{32}, based on the three orthogonal directions x_1, x_2 and x_3 of a right-handed Cartesian coordinate system, are established. They treat the angular momentum of a fluid with density ρ, inside a closed material surface A surrounding a volume V. For this system, partial angular momentum is defined as

$$P_{ij} = \int_V x_i \, \dot{x}_j \, \rho dV \tag{1}$$

Here $i \neq j$. They can take the values 1, 2 and 3. The dot above x denotes total time derivative, which is also written D/Dt. From this definition, it is easily verified that a component of the angular momentum vector can be written

$$L_k = P_{ij} - P_{ji} \,, \tag{2}$$

provided $\{i, j, k\}$ are $\{1, 2, 3\}$, $\{2, 3, 1\}$ or $\{3, 1, 2\}$. Conservation equations for P_{ij} are developed by treating each component of Newton's 2nd law separately. Since Newton's 2nd law treats a specific mass, Lagrangian conservation equations are obtained. They are finally transformed to Eulerian form for an incompressible fluid.

Initially, Newton's 2nd law is adopted on an infinitesimal mass dm with density ρ and volume dV inside a closed surface dA. The external forces are split into gravity forces and surface stresses. For the sake of simplicity the x_3 axis is defined vertical, so that the acceleration of gravity g is in the negative x_3 direction. Then the j^{th} component of Newton's 2nd law reads

$$-\delta_{j3} g\rho \, dV + dF_j = \ddot{x}_j \rho \, dV \,. \tag{3}$$

The Kronecker delta $\delta_{ij} = 1$ when $i = j$. Else $\delta_{ij} = 0$. The first term of (3) is the force of gravity, while dF_j is the component of the stress forces on the surface of dV in the direction of $\mathbf{i}_j$, which is the unit vector along the x_j axis.

Now, (3) is multiplied by x_i:

$$-\delta_{j3} x_i g\rho dV + x_i dF_j = x_i \ddot{x}_j \rho dV \,. \tag{4}$$

By definition, the mass of a material volume $dm=\rho\, dV$ is constant. Hence the right hand side of (4) can be divided into two terms:

$$x_i \ddot{x}_j \rho dV = \frac{D}{Dt}\left(x_i \dot{x}_j \rho dV\right) - \dot{x}_i \dot{x}_j \rho dV \,. \tag{5}$$

The term in the brackets is denoted dP_{ij}, as it is an infinitesimal part of P_{ij} defined in (1). When (5) is inserted into (4), it reads

$$\frac{D}{Dt} dP_{ij} = \dot{x}_i \dot{x}_j \rho \, dV - \delta_{j3} x_i g\rho dV + x_i dF_j \,. \tag{6}$$

The stress force component acting on a small part da of the surface dA of dV, is

$$df_j = \mathbf{i}_j \cdot \sigma \cdot \mathbf{n}\, da\,, \qquad (7)$$

where σ is the stress tensor, $\mathbf{i}_j$ the unit vector in the direction of the x_j axis and $\mathbf{n}$ a unit vector normal to da that is pointing out of dV. By integrating over dA, Gauss' theorem implies that the j^{th} component of the surface forces on dV is obtained:

$$dF_j = \mathbf{i}_j \cdot (\nabla \cdot \sigma) dV\,. \qquad (8)$$

The integration sign is omitted in (8), since dV is infinitesimal so that the integrand can be regarded as a constant.

In order to be used in (6), (8) must be multiplied by x_i. Afterwards, the term on the right hand side is divided into two terms:

$$x_i dF_j = \nabla \cdot (x_i \mathbf{i}_j \cdot \sigma) dV - (\mathbf{i}_j \cdot \sigma) \cdot \nabla x_i dV\,. \qquad (9)$$

Hence the integral of the forces (to the left) equals the integrals over V (to the right). By applying Gauss' theorem on the first term on the right hand side,

$$\int_{F_j} x_i dF_j = \int_A (x_i \mathbf{i}_j \cdot \sigma) \cdot \mathbf{n} dA - \int_V (\mathbf{i}_j \cdot \sigma) \cdot \nabla x_i dV\,. \qquad (10)$$

On the right hand side of (10), the first term is the moment of the component of the surface stresses in the direction of $\mathbf{i}_j$. In the second term, $\nabla x_i \equiv \mathbf{i}_i$. Hence the integrand is σ_{ij}. Since $i \neq j$, the general stress tensor, as given in Landau and Lifshitz (1966) eqns. (15.2) and (15.3), can for this purpose be reduced to

$$\sigma_{ij} = \mu \left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i} \right) \qquad (i \neq j)\,, \qquad (11)$$

where μ is the coefficient of dynamic viscosity.

In order to avoid confusion, please beware that the omission of the stress terms due to compression does not imply that the Lagrangian theory is restricted to incompressible fluids, but that the compression terms do not apply to the transfer of angular momentum.

After integrating (6) and exchanging the last term of it by the right hand side of (10), (6) reads

$$\dot{P}_{ij} = \int_V \dot{x}_i \dot{x}_j \rho\, dV - \int_V \delta_{j3} x_i g \rho dV + \int_A (x_i \mathbf{i}_j \cdot \sigma) \cdot \mathbf{n} dA - \int_V \sigma_{ij} dV \qquad (12)$$

Term 2 and 3 on the right hand side, give the total moment of the external force components in the direction of $\mathbf{i}_j$. By denoting the external moment M_{ij}, and the two remaining terms Θ_{ij}, a conservation equation for partial angular momentum on Lagrangian form is obtained as

$$\dot{P}_{ij} = M_{ij} + \Theta_{ij}\,. \qquad (13)$$

Here Θ_{ij} is found by inserting for σ_{ij} from (11) into the two remaining terms of (12). By exchanging $\dot{x}_i$ by u_i,

$$\Theta_{ij} = \int_V \left[u_i u_j \rho - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV\,. \qquad (14)$$

So Θ_{ij} is a function of viscous stresses and Reynolds stresses.

By inserting (13) into (2), then

$$\dot{L}_k = M_{ij} - M_{ji}\,. \qquad (15)$$

This is the ordinary conservation equation for angular momentum, since the right hand side is the moment of external forces. The two Θ_{ij} cancel, because Θ_{ij} is symmetrical. It implies that they are internal transfer terms for angular momentum between a pair of P_{ij}. So shear forces of one sort or another, transfer angular momentum between a pair of partial angular momenta, while only normal pressures contributes to the energy transfer terms given in Naeser (2002).

Please beware that Θ_{ij} is also nonzero for a laminar flow that is not parallel to any axis. For such cases, it reflects the increase or decrease of the partial angular momenta as the center of mass is moving. The rate of change of a pair of transfer terms are equal under such circumstances. Therefore the angular momentum is conserved as it should when external moments are absent.

Eulerian Conservation Equation for Incompressible Fluids: In order to get (13) on Eulerian form, the left hand side of it is split into a partial time differentiation term and a convective term. By defining

$$p_{ij} = \rho x_i \dot{x}_j \qquad (16)$$

the following equation is valid for the fluid inside the volume V surrounded by a fixed surface A:

$$\dot{P}_{ij} = \int_V \frac{dp_{ij}}{dt} dV\,. \qquad (17)$$

For an incompressible fluid

$$\int_V \frac{dp_{ij}}{dt} dV = \int_V \frac{\partial p_{ij}}{\partial t} dV + \int_V \mathbf{v} \cdot \nabla p_{ij} dV\,. \qquad (18)$$

where $\mathbf{v}$ is the velocity vector. Since $\nabla \cdot \mathbf{v} = 0$, (17) and (18) imply

$$\dot{P}_{ij} = \int_V \frac{\partial p_{ij}}{\partial t} dV + \int_A \mathbf{n} \cdot \mathbf{v} p_{ij} dA\,. \qquad (19)$$

where and $\mathbf{n}$ is a unit vector normal to dA that is pointing out of V. We define

$$Q_{ij} = -\int_A \mathbf{n} \cdot \mathbf{v} p_{ij} dA\,. \qquad (20)$$

It is the flow of partial angular momentum to the system through A.

According to this, the conservation equation for partial angular momentum on Eulerian form for a incompressible fluid is obtained by exchanging the left hand side of (13) with the expression from (19) and (20):

$$\frac{\partial P_{ij}}{\partial t} = M_{ij} + \Theta_{ij} + Q_{ij} \,. \tag{21}$$

By this, the equations that govern the conservation of partial angular momentum are established. In the following section, some consequences of partial angular momentum conservation are shown using simple methods.

Consequences of Water Wave Dissipation: A two-dimensional situation is studied. Regular, deep water Stokes (1847) waves that are traveling in the direction of $\mathbf{i}_1$ are treated. To the second order, for waves with circular frequency ω and wave height H, the energy per unit surface area is

$$E(\omega) = \tfrac{1}{8}\rho g H^2 \,, \tag{22}$$

and, according to Naeser (1979), the angular momentum (wave spin) per unit surface area is

$$S_2(\omega) = \frac{E(\omega)}{2\omega} \tag{23}$$

relative to a moment point at the mean water level. Index 2 is included in (23) to show that the wave spin is a vector in the direction of $\mathbf{i}_2$.

In addition to wave spin, a horizontal current $U(x_3)$ may contribute to the angular momentum, while contributions from turbulence are neglected. Hence we adopt an Eulerian system containing the water in a rectangular part of an infinitely long wave channel, with waves traveling in one direction, while the currents may move in the same direction as the waves at some depths, and in the opposite direction at other depths.

In the following it is shown how waves and currents contribute to P_{13} and P_{31}. We start with the currents. As they are horizontal, they only contribute to P_{31}. If the current profile is known, P_{31} can be calculated. But there is no need for that here, apart from recognizing that $P_{31} < 0$ for a flow in the direction of $\mathbf{i}_1$, since the moment point is located at the mean water level.

The simplest way to calculate the mean value of the partial angular momentum of the waves per unit surface area due to the vertical velocity component, is by a Lagrangian approach. It is found – by integrating through a wave period T – the contribution from all fluid elements that are below a wave crest at $x_1 = 0$ when $t = 0$:

$$P_{13} = \rho \int_0^T \int_{-h}^0 w(x_3)[x(x_3) + U_s(x_3)t]dx_3\,dt \,. \tag{24}$$

Here $x(x_3)$ is the x_1-coordinate relative to the center of the circular paths of the fluid elements, $u(x_3)$ and $w(x_3)$ the sinusoidal horizontal and vertical velocity components, h the water depth and $U_s(x_3)$ the Stokes drift which is a second order horizontal flow caused by the waves. Further, as deep water wave theory is adopted, h is supposed to be large enough to fulfill the deep water requirement. To the second order of deep water Stokes waves,

$$P_{13} = \rho \int_0^T \int_{-h}^0 \tfrac{1}{2}\omega H e^{kx_3} \sin \omega t \times [\tfrac{1}{2} H e^{kx_3} \sin \omega t + \dot{U}_s(x_3)t]dx_3 dt \tag{25}$$

where k is the wave number. Second order harmonic terms are omitted, since they merely contribute to fourth order terms. Integration of (25) yields

$$P_{13} = \frac{E(\omega)}{2\omega} \,. \tag{26}$$

According to (2), (23) and (26), the other partial angular momentum of the waves is

$$P_{31} = P_{13} - S_2 = \frac{E(\omega)}{2\omega} - \frac{E(\omega)}{2\omega} = 0 \,. \tag{27}$$

It vanishes because the contribution from the Stokes drift cancels the contribution from the linear terms. Hence the angular momentum of the system of waves and currents is separated, so that P_{13} includes all wave spin, while P_{31} includes all angular momentum of currents. Therefore the transfer term Θ_{13} equals the transfer of angular momentum between waves and currents.

Through the vertical boundaries in either end, the transport of partial angular momentum due to vertical velocities vanishes, because u and w are 90° out of phase:

$$Q_{13} = -\rho \int_0^T \int_{-h}^0 u(x_3) X w(x_3) dx_3\, dt = 0 \,. \tag{28}$$

Here X is the x_1-coordinate of the boundary.

The next to be calculated is the transfer term Θ_{13}. The orbital terms of the Stokes waves do not contribute to it. So, only currents may contribute. A current $U(x_3)$, whether it is generated by wind or not, implies a transfer per unit surface area that according to (14) is

$$\Theta_{13} = -\int_{-h}^0 \mu \frac{dU(x_3)}{dx_3} dx_3 = -\mu U(0) \,, \tag{29}$$

because the boundary condition at the bottom implies that $U(-h) = 0$.

According to (29), a velocity gradient of U implies amplification of vertical velocities. Hence an internal frictional loss of angular momentum from the horizontal current is transferred to angular momentum of vertical velocities. Below the depths of significant wave motion this implies interactions with turbulence, but nearer the surface, amplification of waves is an alternative. In particular the wind generated surface current is a good candidate for transfer of angular momentum to waves by Θ_{13}, as shown in Naeser (2001). In this way a source to the wave spin of wind-generated waves is obtained. When turbulence is taken into consideration, the $u_i u_j$ term of (14) will contribute too. Hence when (29) is interpreted, μ should be treated in the usual way as "turbulent viscosity".

It is worth to notice that when waves are absent, external wind forces act through the moment point, since it is located at the surface. But regardless the absence of torques from the wind, the current that the wind generates obtain angular momentum. Therefore a flow has to be added in order not to violate (15). Waves is one solution to the problem, as the wave spin is opposite to the currents' contribution. A weak return flow at large depths may solve the problem too. So which of the alternatives is correct? Conventional angular momentum does not distinguish between the two possibilities. But as we have seen, partial angular momentum does, as the shear stresses generate vertical velocities.

When the wind blows, one would imagine that angular momentum can be fed into the waves through the surface by M_{13}, i.e. directly from the air to P_{13} of the waves. But the mean value of the vertical surface forces cancels gravity and bottom pressure everywhere, whether a wind is blowing or not. So $M_{13} = 0$ on an average. Therefore all wave spin has to be transferred from the currents to the waves by Θ_{13}. Hence the importance of shear currents, as suggested by Valenzuela (1976) and Belcher *et al.* (1994), is demonstrated.

In the absence of shear currents, (13) and (26) imply that

$$\sum_{\omega} \frac{E(\omega)}{\omega} = \text{constant} \tag{30}$$

provided the directional distribution does not change. Hence when the wave energy is reduced by internal dissipation, the remaining wave energy should be expected to be downshifted to lower frequencies in order to keep the sum constant. Experimental evidences of these phenomena are shown in Naeser (2000), where these processes are described in more detail.

Acknowledgement

I would like to thank Dr. John T. Conway for his help during a lot of discussions.

Conclusion

I would like to point out the difference between the transfer terms for angular momentum and energy, the latter presented in Naeser (2002). We have seen that angular momentum utilizes shear forces (Reynolds stresses included) for internal transfer, while energy utilizes normal pressures for their internal transfer. So in this respect they are complementary and together they utilize the entire stress tensor for transfer purposes. When instabilities grow, both normal and shear forces are present. Hence both energy transfer and angular momentum transfer take place. Therefore a study the angular momentum in fluid flow appears to be just as important as the study of energy. Both sets of equations have to be fulfilled, and they may give different information as seen in the examples with the waves.

In the section where waves are studied, the transfer term is central. In the absence of a surface current it was found that the transfer term vanishes when μ is constant in space. Then the wave spin of water waves is constant. Thereby the downshifting, that is described in Naeser (1979; 1981 and 2000), is finally proved to be a dissipative phenomenon explained by conservation of partial angular momentum.

The importance of a shear flow during wave generation by wind is demonstrated without the complicated mathematical tools that usually dominate such analyses. Since the wave spin has to be transferred to the waves from the horizontal current, the integral given in (29) gives the rate of change of wave spin, when μ is interpreted as a sum of laminar viscosity and "turbulent viscosity". From (23) the relationship between energy and spin is given for surface waves. Hence when the input of wave spin is known, the growth of the waves at a given frequency can be calculated.

In conclusion: The partial angular momentum and the partial energy provide new and powerful means to solve fluid dynamical problems. I leave the challenge of finding other applications and better methods to the reader.

References

Belcher, S. E., J. A. Harris and R. L. Street, 1994. Linear Dynamics of Wind Waves in Coupled Turbulent Air-Water Flow, Part 1. Theory. J. Fluid. Mech., 271, 119–151.

Landau, L. D. and E. M. Lifshitz, 1966, Fluid Mechanics, Pergamon Press.

Stokes, G. G., 1847. On the Theory of Oscillatory Waves, Trans. Cambridge Phil. Soc., 8, 441 – 455

Naeser, H., 1979, Why Ripples Becomes Swells and Swells in Shallow Water Decay Rapidly, Geophysical and Astrophysical Fluid Dynamics, 13, 335 – 345.

Naeser, H., 1981, A Theory for the Evolution of Wind-Generated Gravity-Wave Spectra due to Dissipation, Geophysical and Astrophysical Fluid Dynamics, 18, 75 – 96.

Naeser, H., 2000. Energy Accumulation in Ocean Waves, Proc. 3rd European Wave Energy Conf. 1998. 2, 296 – 303

Naeser, H., 2001. Energy Accumulation in Ocean Waves by Wind-Supported Capillary Waves, Proc. 4th European Wave Energy Conf. 2000, 80 – 87.

Naeser, H., 2002. Conservation of "Partial Energy", with an Application on Water Wave Dissipation, Pakistan J. of Applied Sciences, 2: xx – xx.

Valenzuela, G. R., 1976. The Growth of Gravity-Capillary Waves in a Coupled Shear Flow, J. Fluid. Mech., 76, 229 – 251.