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Matrix Representation of the Tetrahedron Groups

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Abstract: We present the method by which one can get the matrix representation of the tetrahedron groups introduced in Lannér, (1950)

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Introduction

Lannér, (1950) and Best, (1971) listed the nine non-congruent tetrahedra. With each of these tetrahedra one can associate a subgroup of the whole group of isometries, $P\Gamma L(2, C)$, of H^3, generated by reflections in faces of each tetrahedron. This subgroup we call it the tetrahedron group.

In $P\Gamma L(2, C)$, we find the matrices representing the generators of each tetrahedron group through the consideration of its action on H^3. This matrix representation is found through finding the orientation preserving subgroup first and then present the matrices representing the reflections.

The Three – Dimensional Geometry:

Let $H^3 = \{(x, y, w) \in R^3 : w > 0\}$.

We my regard H^3 as a subspace of the space

$H = \{x + yi + wj + tk : x, y, w, t \in R\}$ of quaternion

with the usual multiplication,

$i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

The sub ring $R + Ri$ is isomorphic to C and will be identified with C. Then

$H^3 = \{\zeta = z + wj : z \in C, w \in r, w > 0\}$.

Consider the group $G = PSL(2,C)$ acting on H^3 as follows.

Let

$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $(\alpha, \beta, \gamma, \delta \in C, \alpha\delta - \beta\gamma = 1)$

be an element of G and let $\zeta \in H^3$. Then the action of G on H^3 is given by:

$$g : \varsigma \mapsto \frac{\alpha\varsigma + \beta}{\gamma\varsigma + \delta} \quad (1)$$

It is clear that the the image of ς is also in H^3 (Harvey, 1977)

We can prove the following lemma.

Lemma 1: G is homeomorphic to the product of stabilizer of a point of H^3 and the subgroup G_0 of G consisting of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \quad \alpha \in R, \quad \alpha > 0, \quad \beta \in C.$$

Proof The action of elements of G_0 on j is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : j \mapsto \alpha\beta + \alpha^2 j \in H^3$$

Therefore H^3 is a G-orbit, and hence G is transitive in its action on H^3.

In fact, the map $\phi : G_0 \to H^3$ given by $\phi(g) = g(j)$ defines a homeomorphism between G_0 and H^3. We consider now the stabilizer of j in G. $\begin{pmatrix} \alpha & \beta \\ \gamma & \alpha^{-1} \end{pmatrix} \in G,$ and $(\alpha\, j + \beta)(\alpha\, j + \delta)^{-1} = j$ means $\gamma = -\overline{\beta}, \delta = \overline{\alpha}.$

Therefore, the stabilizer of j is the group $\left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in C, \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \right\}$

Which is the group PSU(2,C) of all unitary matrices.

The map $h : S^3 \to PSU(2,C)$ given by $h(\alpha + \beta\, j) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ for all $\alpha + \beta\, j \in S^3$, defines a homeomorphism between S^3 and $PSU(2,C)$.

Since G is transitive in its action on H^3, then the stabilizer of any element ς of H^3 is conjugate to the stabilizer of j.

Let $g \in G$, then

$g\, j = \phi^{-1}(g(j))j = \psi(g)\, j$, say. Therefore, $(\psi(g))^{-1} g\, j = j$, that is to say $(\psi(g))^{-1} g \in PSU(2, C)$.

Now $\psi(g) \in G_0$, then $g = \psi(g)(\psi(g))^{-1} g$.

This last formula sets up a homeomorphism between G and the product of G_0 and $PSU(2,C)$.

Therefore G is the whole group orientation preserving isometries of H^3 because it contains the whole orthogonal subgroup acting on tangent space at j, and also it is transitive on H^3.

Consider the transformation

$t:\ \zeta = z + wj \mapsto \bar{z} + wj$. Consider t as

$(x, y, w) \mapsto (x, -y, w)$,

so t is an isometry and it reverses orientation.
Let u be any other orientation reversing transformation, then ut is an orientation preserving element and therefore it is in $PSL(2, C)$.
Adjoin t to $PSL(2, C)$ we get the whole group of isometries $P\Gamma L(2, C)$.

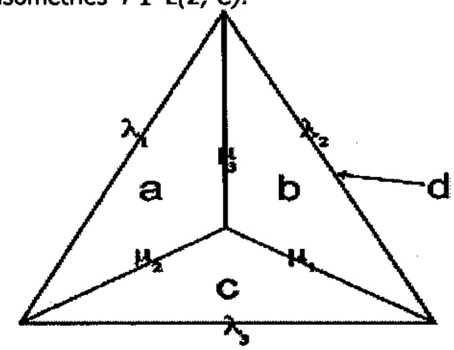

Fig.1:The Hyperbolic Tetrahedron Groups

The Hyperbolic Tetrahedron Groups: Lannér,(1950) has shown that there are exactly nine non-congruent hyperbolic tetrahedra with all dihedral angles equal to an integer sub multiple of π

Let π/λ_i and $\pi/\mu_i, i = 1,2,3,$ be the dihedral angles at opposite edges of the tetrahedron, where π/λ_i, $i = 1,2,3$ are the angles at the edges of a face.

We list the nine non-congruent tetrahedra by writing $[\lambda_1, \lambda_2, \lambda_3 : \mu_1, \mu_2, \mu_3]$ in each case.

T1 = [2, 2, 3 : 3, 5, 2] , T2 = [2, 2, 3 : 2, 5, 3] ,
T3 = [2, 2, 4 : 2, 3, 5] , T4 = [2, 2, 5 : 2, 3, 5] ,
T5 = [2, 3, 3 : 2, 3, 4] , T6 = [2, 3, 4 : 2, 3, 4] ,
T7 = [2, 3, 3 : 2, 3, 5] , T8 = [2, 3, 4 : 2, 3, 5] ,
T9 = [2, 3, 5 : 2, 3, 5] .

With each tetrahedron we associate a group of isometries of H^3, call it the tetrahedral group, generated by the reflections in its faces. Because of the angle condition, this group is discrete in each case listed and has this presentation:

$$< a,b,c,d\ :\ a^2=b^2=c^2=d^2=(bc)^{\mu_1}=(ca)^{\mu_2}=(ab)^{\mu_3}= (ad)^{\lambda_1}=(bd)^{\lambda_2}=(cd)^{\lambda_3}=1 >$$

Where a, b, c and d are reflections in faces of tetrahedron.

The orientation preserving subgroup has presentation:

$$< x_1, x_2, x_3\ :\ x_1^{\lambda_1} = x_2^{\lambda_2} = x_3^{\lambda_3} = (x_2x_3^{-1})^{\mu_1} = (x_3x_1^{-1})^{\mu_2} = (x_1x_2^{-1})^{\mu_3} = 1 >$$

Where x_1 = ad, x_2 = bd, x_3 = cd.

Matrix representation of the tetrahedral groups: Now T1, ...,T9 are to denote not the tetrahedra , but the corresponding tetrahedral groups.
We find matrices representing the orientation preserving subgroup of the tetrahedral group T1 ,

that is, $< x_1, x_2, x_3 : x_1^2 = x_2^2 = x_3^3 = (x_2x_3^{-1})^3 = (x_3x_1^{-1})^5 = (x_1x_2^{-1})^2 = 1 >$.

Let X_1, X_2, X_3 be the matrices representing x_1, x_2, x_3 respectively . The action of X_i is defined by (1). Each X_i has determinant 1 and x_i's being rotations through $\frac{2\pi}{n_i}$, the traces of X_i 's will be equal to $\pm 2\cos(\pi/n_i)$. So the matrices Xi are found by solving the equations obtained from the determinants and traces of $X_1, X_2, X_3, X_2X_3^{-1}, X_3X_1^{-1}$, and $X_1X_2^{-1}$.
From these matrices one can find the matrices associated with the reflections a, b, c and d of the tetrahedron group T1.

If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is a matrix associate of a reflection, then it is associated with the orientation reversing isometry

$$\zeta \mapsto (p\zeta^j + q)(r\zeta^j + s)^{-1} \qquad p,q,r,s \in C \quad (2)$$

Where $\zeta^j = t\zeta = j\,\zeta\,j^{-1} = -j\,\zeta\,j.$

For (2) to be associated with a reflection, we have

$$ps - qr = -1, \quad s = -\bar{p}.$$

If A and B are two matrix associates of reflections given in this way, then the orientation preserving isometry which is the product of these two reflections will be associated in the ordinary way (1) with the matrix $A\bar{B}$ (where the bar denotes complex conjugation).
So having the matrices X_1, X_2, X_3 known, then if A, B, C, D are the matrix associates of the reflections a, b, c, d respectively, then we have

T1 = $< a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^3 = (ca)^5 = (ab)^2 = (ad)^2 = (bd)^2 = (cd)^3 = 1 >$

and can be represented by:

$$A=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} \frac{1}{2}-\frac{\sqrt{5}+1}{4}i & -\frac{1}{2}-\frac{\sqrt{\sqrt{5}-1}}{2\sqrt{2}} \\ -\frac{1}{2}+\frac{\sqrt{\sqrt{5}-1}}{2\sqrt{2}} & -\frac{1}{2}-\frac{\sqrt{5}+1}{4}i \end{pmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A similar method can be used to find the matrices associated with the generators of T2 – T9, and we list them below.

T2 = < *a, b, c, d, : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^5$=$(ab)^3$=$(ad)^2$ = $(bd)^2$ = $(cd)^3$ = 1* >

$$A=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad B=\begin{pmatrix} -\frac{i}{2} & 3 \\ \frac{1}{4} & -\frac{i}{2} \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{1}{2}-\frac{\sqrt{5}+1}{4}i & -\frac{\sqrt{5}+1}{2}+\sqrt{2\sqrt{5}-3} \\ -\frac{\sqrt{5}+1}{24}-\frac{\sqrt{2\sqrt{5}-3}}{12} & -\frac{1}{2}-\frac{\sqrt{5}+1}{4}i \end{pmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

T3 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^5$=$(ab)^5$=$(ad)^2$ = $(bd)^2$ = $(cd)^4$ = 1* >

$$A=\begin{pmatrix} \frac{\sqrt{5}+1}{4}+\frac{i}{2} & \sqrt{2}-\sqrt{\sqrt{5}-1} \\ \frac{1}{8}\left(\sqrt{2}+\sqrt{\sqrt{5}-1}\right) & -\frac{\sqrt{5}+1}{4}+\frac{i}{2} \end{pmatrix},\quad B=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad D=\begin{pmatrix} \frac{i}{\sqrt{2}} & -2 \\ -\frac{1}{4} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

T4 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$=$(ab)^5$= $(ad)^2$ = $(bd)^2$ = $(cd)^5$ = 1* > .

$$A=\begin{pmatrix} -\frac{\sqrt{5}+1}{4}i & -2+\frac{\sqrt{5\sqrt{5}-7}}{\sqrt{2}} \\ -\frac{1}{5\sqrt{5}}-\frac{\sqrt{5\sqrt{5}-7}}{10\sqrt{2}-2\sqrt{10}} & -\frac{\sqrt{5}+1}{4}i \end{pmatrix},\quad B=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$C=\begin{bmatrix} \frac{\sqrt{5}+1}{4} & \frac{5-\sqrt{5}}{2} \\ \frac{1}{4} & -\frac{\sqrt{5}+1}{4} \end{bmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

T5 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$=$(ab)^4$=$(ad)^2$ = $(bd)^3$ = $(cd)^3$ = 1* >

$$A=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad B=\begin{pmatrix} \frac{1}{2}+\frac{i}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}+\frac{i}{\sqrt{2}} \end{pmatrix},$$

$$C=\begin{pmatrix} \frac{1}{2}+\frac{i}{2} & \frac{1}{2}\sqrt{1+2\sqrt{2}}-\frac{1}{2}\left(1+\sqrt{2}\right) \\ -\frac{1}{2}\sqrt{1+2\sqrt{2}}-\frac{1}{2}\left(1+\sqrt{2}\right) & -\frac{1}{2}+\frac{i}{2} \end{pmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

T6 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$=$(ab)^4$= $(ad)^2$ = $(bd)^3$ = $(cd)^4$ = 1* > .

$$A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad B=\begin{pmatrix} -\frac{1}{\sqrt{2}}+\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}}+\frac{i}{2} \end{pmatrix},$$

$$C=\begin{pmatrix} -\frac{1}{2}+\frac{i}{\sqrt{2}} & -\sqrt{2}+\frac{\sqrt{7}}{2} \\ -\sqrt{2}-\frac{\sqrt{7}}{2} & \frac{1}{2}+\frac{i}{\sqrt{2}} \end{pmatrix},\quad D=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

T7 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$ = $(ab)^5$ =$(ad)^2$ = $(bd)^3$ = $(cd)^3$ = 1* >

$$A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad B=\begin{pmatrix} -\frac{\sqrt{5}+1}{4}+\frac{i}{2} & -\frac{3+\sqrt{5}}{4}-\frac{\sqrt{5\sqrt{5}+1}}{2\sqrt{2}} \\ -\frac{3+\sqrt{5}}{8}+\frac{\sqrt{5\sqrt{5}+1}}{4\sqrt{2}} & \frac{\sqrt{5}+1}{4}+\frac{i}{2} \end{pmatrix},$$

$$C=\begin{pmatrix} -\frac{1}{2}+\frac{i}{2} & 1 \\ \frac{1}{2} & \frac{1}{2}+\frac{i}{2} \end{pmatrix},\quad D=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

T8 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$ = $(ab)^5$ =$(ad)^2$ = $(bd)^3$ = $(cd)^4$ = 1* >

$$A=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad B=\begin{pmatrix} \frac{1}{2}\frac{\sqrt{5}+1}{4}i & \frac{\sqrt{5}+1}{4}\frac{1}{\sqrt{2}}+\frac{\sqrt{\sqrt{10}+\sqrt{5}+\sqrt{2}+2}}{2} \\ \frac{\sqrt{5}+1}{4}\frac{1}{\sqrt{2}}\frac{\sqrt{\sqrt{10}+\sqrt{5}+\sqrt{2}+2}}{2} & \frac{1}{2}\frac{\sqrt{5}+1}{4}i \end{pmatrix},$$

$$C=\begin{pmatrix} \frac{1}{\sqrt{2}}-\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}}-\frac{i}{2} \end{pmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

T9 = < *a, b, c, d : a^2 = b^2 = c^2 = d^2 = $(bc)^2$ = $(ca)^3$ = $(ab)^5$ =$(ad)^2$ = $(bd)^3$ = $(cd)^5$ = 1* >

$$A=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad B=\begin{pmatrix} \frac{1}{2}-\frac{\sqrt{5}+1}{4}i & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{1}{2}-\frac{\sqrt{5}+1}{4}i \end{pmatrix},$$

$$C=\begin{pmatrix} \frac{\sqrt{5}+1}{4}-\frac{i}{2} & -\frac{3+\sqrt{5}}{2}+\frac{\sqrt{7\sqrt{5}+5}}{\sqrt{2}\left(\sqrt{5}-1\right)} \\ -\frac{3+\sqrt{5}}{2}-\frac{\sqrt{7\sqrt{5}+5}}{\sqrt{2}\left(\sqrt{5}-1\right)} & -\frac{\sqrt{5}+1}{4}-\frac{i}{2} \end{pmatrix},\quad D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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