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Determination of Time Required to Empty a Reservoir Using Dimensional Analysis

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Abstract: The main goal of this paper is to demonstrate the theoretical and practical use of dimensional analysis, π-theorem, to formulate a physical model of fluid flow from a reservoir. Such models are essential in chemical, mechanical, and reservoir engineering, specially, to find the time required to empty a tank or a reservoir. This work showed that dimensional analysis reduced the number of experimental variables to be correlated (measured), and pointed out the best experimental approach to the problem. Use of dimensional analysis results in significant saving of time, efforts, and expense in experimental investigation and correlation.

Keywords: Dimensional Analysis, Emptying a Tank, Reservoir, Time, π-Theorem, Fluid, Flow

Introduction

Dimensional analysis is a mathematical tool that enables engineers to save considerable time in planing experiments and correlating their results. Application of the dimensional analysis to a situation requires the knowledge of only the variables believed to be involved and their dimensions (Buckingham, 1915).

Buckingham π-method was used to solve different fluid problems (Bird, 1960. and Victor, 1986). In this study we used the π-theorem to design an experiment to measure the time required to empty a tank or a reservoir. The main idea of dimensional analysis lies in the fact that, the dependence of any physical parameter α on a group of independent parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$, can be written in the following form:

$$\alpha = f(\alpha_1, \alpha_2, \cdots, \alpha_n) \qquad (1)$$

Numerical values of parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$, in general, are dimensional values and depend on the unit of measurement. The main theory of dimensional analysis (π-theorem) can be stated as follows; the dimensionally homogeneous equation (1) can be reduced to a relationship among a complete set of dimensionless products $(n+1-k)$, where k is the number of dimension-independent variables of $\alpha_1, \alpha_2, \cdots, \alpha_n$.

If dimension-independent variables are $\alpha_1, \alpha_2, \cdots, \alpha_k$, where $k \le n$, then the dimensions of the remaining variables can be expressed through dimensions of independent variables with the help of the following dimensionless equation:

$$\left.\begin{aligned} [\alpha] &= [\alpha_1]^{m_1}[\alpha_2]^{m_2}\cdots[\alpha_k]^{m_k} \\ [\alpha_{k+1}] &= [\alpha_1]^{n_1}[\alpha_2]^{n_2}\cdots[\alpha_k]^{n_k} \\ [\alpha_n] &= [\alpha_1]^{l_1}[\alpha_2]^{l_2}\cdots[\alpha_k]^{l_k} \end{aligned}\right\} \qquad (2)$$

Where $m_i, n_i, l_i, (i = 1 \cdots k)$ are real numbers. Dimensionless products about which we are talking in π-theorem have the following form:

$$\left.\begin{aligned} \pi &= \frac{\alpha}{\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_k^{m_k}} \\ \pi_1 &= \frac{\alpha_{k+1}}{\alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_k^{n_k}} \\ &\cdots \\ \pi_{n-k} &= \frac{\alpha_n}{\alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_k^{l_k}} \end{aligned}\right\} \qquad (3)$$

Now, equation (1) can be written as:

$$\pi = F(\pi_1, \pi_2, \cdots, \pi_{n-k}) \qquad (4)$$

It is clear that the last equation above is independent of the system of units.

Similarity and Modeling Approach:

The importance of dimensional analysis came from the creation of the rules of similarity between physical phenomena, which have the same characteristics. This enables the observer of a natural phenomenon to model it with a similar one but in a pilot plant (experimental) scale.

Before proceeding further, the necessary and sufficient conditions for similarity of two phenomena were defined. Two phenomena are similar, if by given parameters of one of them similar parameters of the other phenomenon can

be determined by simple calculations. Each group of similar phenomena is usually called a model of other phenomenon of the same group.
Assume a phenomenon for which a variable α is defined by equation (1). The model of this phenomenon is formulated by the dependence of the same physical variables α' on the same parameters with values $\alpha'_1, \alpha'_2, \cdots, \alpha'_n$, then:

$$\alpha' = f(\alpha'_1, \alpha'_2, \cdots, \alpha'_n) \quad (5)$$

According to π-theorem, dependence f can be written in dimensionless form as:

$$\pi = F(\pi_1, \pi_2, \cdots, \pi_{n-k}) \quad (6)$$

$$\pi' = F(\pi'_1, \pi'_2, \cdots, \pi'_{n-k}) \quad (7)$$

Where k is the number of dimension-independent parameters among $\alpha_1, \alpha_2, \cdots, \alpha_n$, and π_i is determined by equation (3).
Equations (6) and (7) show that, regardless the difference in numerical values between α_1 and α'_1, α_2 and α'_2, etc., the values of all dimensionless products might be the same, so:

$$\left.\begin{aligned} \pi_1 &= \pi'_1 \\ \pi_2 &= \pi'_2 \\ &\cdots \\ \pi_{n-k} &= \pi'_{n-k} \end{aligned}\right\} \quad (8)$$

If it is possible to choose the values of parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\alpha'_1, \alpha'_2, \cdots, \alpha'_n$ in such a way that equation (8) is satisfied, then we can write:

$$\pi = \pi' \quad (9)$$

And we can say that one phenomenon models the other.
Satisfaction of equation (8), which is called similarity condition, gives:

$$\frac{\alpha}{\alpha_1^{m_1}\,\alpha_2^{m_2}\cdots\alpha_k^{m_k}} = \frac{\alpha'}{\alpha_1'^{m_1}\,\alpha_2'^{m_2}\cdots\alpha_k'^{m_k}} \quad (10)$$

Then:

$$\alpha = \alpha' \frac{\alpha_1^{m_1}\,\alpha_2^{m_2}\cdots\alpha_k^{m_k}}{\alpha_1'^{m_1}\,\alpha_2'^{m_2}\cdots\alpha_k'^{m_k}} \quad (11)$$

Equation (11) means that if we know the value of α', then we can easily determine the value of α.

Modeling the Process of Fluid Flow from a Reservoir: Assume that a cylindrical tank Fig.1 is filled with viscous incompressible fluid. We need to find the time (t) required to empty this tank.
It is clear that the time required to empty a tank is a function of cylindrical reservoir dimensions: its diameter D, length L, and the diameter of the hole through which the fluid will drain d. Moreover, it's important to know the type of fluid, i.e. its density ρ, dynamic coefficient of viscosity μ, also the characteristic of the force (driving force) which causes the fluid to flow. In this case the driving force is the gravity force. By considering all these factors, we can write:

$$t = f(\rho, \mu, g, D, d, L) \quad (12)$$

Experimentally, the time (t') required to empty a model reservoir can be measured. The model reservoir is similar to the real reservoir but with the lab scale dimensions. The time (t') will be determined by equation (12) but using the following values of defining parameters: ρ', μ', g', D', d', and L', that characterize the model reservoir, so:

$$t' = f(\rho', \mu', g', D', d', L') \quad (13)$$

Analyzing the units of all arguments of function f, we have:

$$\left.\begin{aligned} [\rho] &= M\,L^{-3} \\ [\mu] &= M\,L^{-1}\,T^{-1} \\ [g] &= L\,T^{-2} \\ [D] &= [d] = [L] = L \end{aligned}\right\} \quad (14)$$

It is clear that among the above six parameters three are dimension-independent. Thus it is possible to build or formulate four $(6+1-3=4)$ dimensionless products as:

$$\left.\begin{aligned} \pi &= \frac{t}{\sqrt{D/g}} \\ \pi_1 &= \frac{\sqrt{g\,D}\,D}{\mu/\rho} = \frac{\sqrt{g\,D}\,D}{\nu} \\ \pi_2 &= \frac{d}{D} \\ \pi_3 &= \frac{L}{D} \end{aligned}\right\} \quad (15)$$

Taking this into account, equation (12) can be written as:

$$\left.\begin{aligned} &\pi = F(\pi_1, \pi_2, \pi_3) \\ &\text{or} \\ &\frac{t}{\sqrt{D/g}} = F\left(\frac{\sqrt{g\,D}\,D}{\nu}, \frac{d}{D}, \frac{L}{D}\right) \end{aligned}\right\} \quad (16)$$

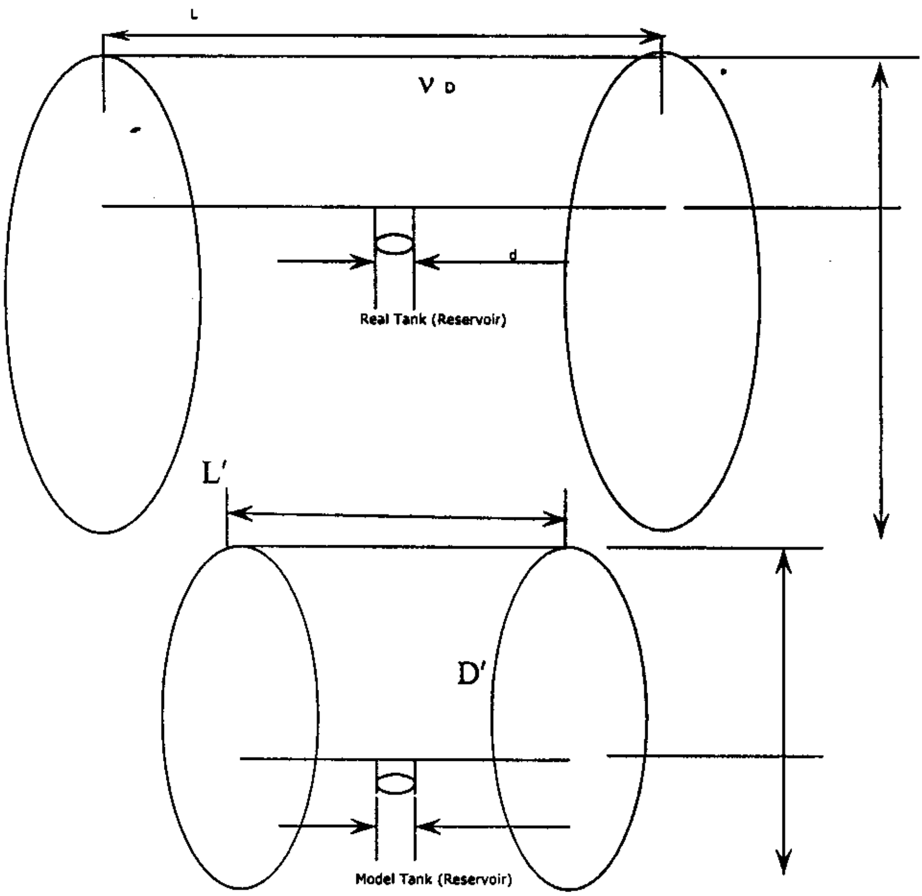

Fig. 1: Schematic Representation of the Real and Model Reservoirs

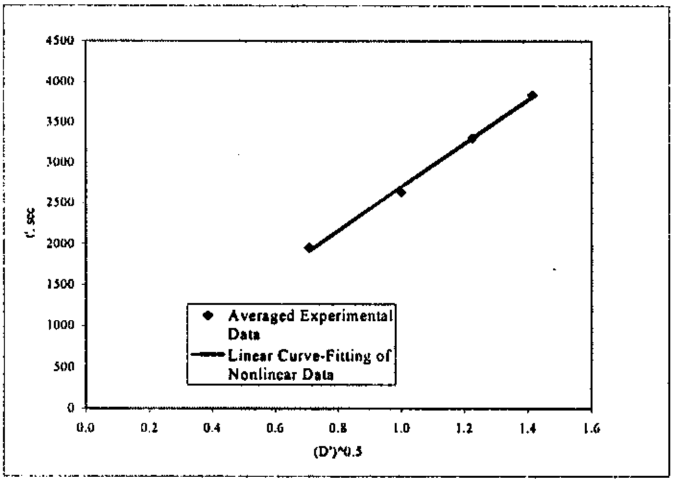

Fig. 2: Experimental Data and Their Linear Curve-Fitting for the Time Required to Drain a Tank of Different Diameters

Similarly, equation (13) becomes:

$$\left.\begin{aligned}&\pi' = F(\pi_1', \pi_2', \pi_3')\\ &\text{or}\\ &\frac{t'}{\sqrt{D'/g}} = F\left(\frac{\sqrt{g\,D'}\,D'}{\nu'}, \frac{d'}{D'}, \frac{L'}{D'}\right)\end{aligned}\right\} \quad (17)$$

The necessary and sufficient condition of similarity between the model and real cases is that:

$$\left.\begin{aligned}&\frac{L}{D} = \frac{L'}{D'}\\ &\frac{d}{D} = \frac{d'}{D'}\\ &\frac{\sqrt{g\,D}\,D}{\nu} = \frac{\sqrt{g\,D'}\,D'}{\nu'}\end{aligned}\right\} \quad (18)$$

The first two conditions above characterize geometrical similarity of two cylindrical reservoirs. The third condition, which can be rearranged in the form:

$$\nu' = \nu\left(\frac{D'}{D}\right)^{\frac{3}{2}} \quad (19)$$

is needed for the selection of the kinematic viscosity of the fluid (ν'), which should be used in modeling. If the condition (18) exists, then, the following equity will be satisfied:

$$\frac{t}{\sqrt{D/g}} = \frac{t'}{\sqrt{D'/g}} \quad (20)$$

And the time required to empty a cylindrical reservoir or tank (t) can be determined with the knowledge of the time required to empty a model reservoir (t') by the following relationship:

$$t = t'\sqrt{D/D'} \quad (21)$$

Experimental measurements of t' will lead to the determination of t by a very simple calculation. Equation (21) can be rewritten as:

$$t' = \frac{t}{\sqrt{D}}\sqrt{D'} \quad (22)$$

Experimentally, we measured the variable t' for several values of another variable D'. A plot of t' versus $\sqrt{D'}$ has a linear behavior Fig.2 with zero intercept. The slope found from the linear fitting equation is equal to the slope of equation (22), so $\text{slope} = t/\sqrt{D}$. Finally, the time required to empty a reservoir or tank can be determined as:

$$t = (slope)\sqrt{D} \quad (23)$$

The last equation above shows that the usage of π-theorem results in a very simple equation which can be used to determine the time required to empty a reservoir of diameter D quite easily.

Conclusions

This study showed that the variables involved in emptying a reservoir are known, while the relationships among them are unknown. By a procedure of dimensional analysis a phenomenon considered was formulated as a relationships among a set of dimensionless groups of the variables. The immediate advantage of this procedure is that considerably less experimentation is required in order to establish relationships among several variables involved. The study showed that it is sufficient to vary the value of only one variable (D') included in the dimensionless group in order to cause the group as a whole to vary over a required range.

The use of the derived equation to find the time required to empty a tank or reservoir results in significant saving of time, efforts, and expense, specially, in the case of emergencies where the time is of special importance.

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