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Legendre Functions Direct Method for Solving Integral Equations

[1]H. Parsian and [2]R. Sabzpoushan
[1]Department of Physics, [2]Department of Electrical and Computer Engineering
University of Bu Ali Sina , Hamadan, Iran

Abstract: A direct method for solving integral equations using Legendre function is presented. An operational matrix introduce for operator of integral equation and it reduce into a set of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key Words: Integral Equation, Operational Matrix, Legendre Function

Introduction

Integral Equations (IE) have received in dealing with various problems of sciences. An integral equation can not be solved exactly always and there is vàrious numerical methods for solving it. The main characteristic of this technique is that it reduces integral equation to a set of algebraic equations, thus greatly simplifying the problem.The approach is based on converting the IE into set of algebraic equation by choosing the answer function as the series of $\Phi(x)=\{\phi_\circ(x),\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x),\phi_5(x),...\}$

and operational matrix L for integral operator L. The elements

$\phi_\circ(x),\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x),\phi_5(x),...$ are the basis function, orthogonal on a certain interval [a,b]. This technique is to be used for variational problems by various basis functions, to applications of Walsh Functions (Chen and Hsiao, 1975), block pulse functions (Hwang and Shih, 1985), Laguerre Functions (Hwang and Shih, 1983), Legendre Functions (Chang and Wang, 1983) and (Razzaghi and Yousefi, 2000), Chebyshev Functions (Horng and Chou, 1985) and Fourier series (Razzaghi, 1988).

In (Parsian, 2002), we introduce a direct computational method for solving Linear Differential Equation(LDE). This method consists of reducing the LDE into a set of algebraic equations by first expanding the candidate function as Legendre Functions with unknown coefficients. In the present paper introduce this method for solving integral equation

$$\int_0^x K(x,t)y(t)dt = a+by(x).$$

In this method the IE reduce into $\mathrm{L}\Phi(x)=0$, in which L matrix of integral operator L and $0=[0,0,0,...,0]^T$ is the column matrix, and finally from solving $\mathrm{L}=0$, exact solution for IE obtained.

Properties of Legendre Functions

Legendre Functions: Legendre Polynomials is the solutions of DE below

$$(1-x^2)y''(x)-2xy'(x)+n(n+1)y(x)=\circ \qquad (1)$$

These are a set of orthogonal functions with respect to the weight function $\omega(x)=1$ on the interval [-1,1] and satisfy the following recursive formulas (Chang and Wang, 1983):

$$xP_n(x)=\left(\frac{n+1}{2n+1}\right)P_{n+1}(x)+\left(\frac{n}{2n+1}\right)P_{n-1}(x) \quad ,n=1,2,3,...$$
$$(1-x^2)P_n'(x)=nP_{n-1}(x)-nxP_n(x)$$
$$P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$$
(2)

Function Approximation: A function $y(x)$ defined over [-1,1] may be expanded as

$$y(x)=\sum_{n=\circ}^{\infty}c_nP_n(x) \qquad (3)$$

where $c_n=\frac{2n+1}{2}<y(x),P_n(x)>$, in which <, > denotes the inner product.

If the infinite series in (3) is truncated, then (3) can be written as

$$y(x)=\sum_{n=\circ}^{N-1}c_nP_n(x) \qquad (4)$$

Matrices Definition

Column Matrices: If we definite C and P(x) as $N\times 1$ matrices below

$$\mathrm{C}=[c_\circ,c_1,c_2,c_3,c_4,...,c_{N-1}]^T$$
$$\mathrm{P}(x)=[P_\circ(x),P_1(x),P_2(x),P_3(x),P_4(x),...,P_{N-1}(x)]^T \qquad (5)$$

then (4) can be written as

$$y(x)=\sum_{n=\circ}^{N-1}c_nP_n(x)=\mathrm{C}^T.\mathrm{P}(x)$$

The numbers and x^m be defined in (Parsian, 2002).

Integration Matrix:

$$\int_0^x x^m y(t)\,dt \equiv C^T.H.(X^m Y - P(0).X^{m^T})$$

Integral of function $\int_0^x y(x)\,dx$ defined in (4) can be obtained as

$$\int_0^x y(x)\,dx = \sum_{n=0}^{N-1} c_n \int_0^x P_n(x)\,dx = \sum_{n=0}^{N-1} \frac{c_n}{2n+1}\left(P_{n+1}(x) - P_{n-1}(x)\right)\Big|_0^x$$

$$= C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).P(x) - C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).P(0).$$

$$= C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).(I - P(0).1^T).P(x)$$

$$= C^T.H.(I - P(0).1^T).P(x)$$

$$= C^T.S.P(x) \qquad (6)$$

where $U_{\frac{1}{2n+1}}$, I^-, I^+ and I(unit matrix) are the $N \times N$ operational matrices that be defined in (Parsian, 2002).

The term $\int_0^x xy(t)\,dt$ can be definite as

$$\int_0^x xy(t)\,dt = \sum_{n=0}^{N-1} xc_n \int_0^x P_n(t)\,dt = \sum_{n=0}^{N-1} \frac{c_n}{2n+1} x\left(P_{n+1}(x) - P_{n-1}(x)\right)\Big|_0^x$$

$$= C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).XP(x) - (C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).P(0))x.$$

$$= C^T.H.(XY - P(0).X^T).P(x) \qquad (7)$$

The term $\int_0^x x^2 y(t)\,dt$ can be definite as

$$\int_0^x x^2 y(t)\,dt = \sum_{n=0}^{N-1} x^2 c_n \int_0^x P_n(t)\,dt = \sum_{n=0}^{N-1} \frac{c_n}{2n+1} x^2\left(P_{n+1}(x) - P_{n-1}(x)\right)\Big|_0^x$$

$$= C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).X^2 P(x) - (C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).P(0))x^2.$$

$$= C^T.H.(X^2 Y - P(0).X^{2^T}).P(x) \qquad (8)$$

The term $\int_0^x x^m y(t)\,dt$ can be definite as

$$\int_0^x x^m y(t)\,dt = \sum_{n=0}^{N-1} x^m c_n \int_0^x P_n(t)\,dt$$

$$= C^T.U_{\frac{1}{2n+1}}.(I^+ - I^-).(X^m P(x) - x^m P(0))$$

$$= C^T.H.(X^m Y - P(0).X^{m^T}).P(x) \qquad (9)$$

Matrix: $\int_0^x t^m y(t)\,dt \equiv C^T.X^m Y.S$

The term $\int_0^x ty(t)\,dt$ can be definite as

$$\int_0^x ty(t)\,dt = \sum_{n=0}^{N-1} c_n \int_0^x tP_n(t)\,dt = \sum_{n=0}^{N-1} c_n \left(\left(\frac{n+1}{2n+1}\right)\int_0^x P_{n+1}(t)\,dt + \left(\frac{n}{2n+1}\right)\int_0^x P_{n-1}(t)\,dt\right)$$

$$= C^T.(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^-).S.P(x)$$

$$= C^T.XY.S.P(x) \qquad (10)$$

and term $\int_0^x t^2 y(t)\,dt$ can be definite as

$$\int_0^x t^2 y(t)\,dt = \sum_{n=0}^{N-1} c_n \int_0^x t^2 P_n(t)\,dt$$

$$= C^T.(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^-)^2.S.P(x)$$

$$= C^T.X^2 Y.S.P(x) \qquad (11)$$

finally

$$\int_0^x t^m y(t)\,dt = \sum_{n=0}^{N-1} c_n \int_0^x t^m P_n(t)\,dt$$

$$= C^T.(U_{\frac{n+1}{2n+1}}.I^+ + U_{\frac{n}{2n+1}}.I^-)^m.S.P(x)$$

$$= C^T.X^m Y.S.P(x) \qquad (12)$$

Direct Method of Legendre Functions: Consider the integral equation

$$y(x) = f(x) + \int_0^x K(x,t)y(t)\,dt$$

where $K(x,t)$ is the kernel of integral equation. With choosing $y(x) = C^T.P(x)$ and to find L operational matrix corresponding to the right hand , we have

$$(C^T - L).P(x) = 0 \qquad (13)$$

Where $0 = [0,0,0,\ldots,0]^T$ is a $N \times 1$ matrix. With in mind that $P(x)$ is a set of orthogonal functions, then $C^T - L = 0$.

Illustrative Examples:

Example 1.

Consider the given IE

$$y(x) = 1 - 2\int_0^x ty(t)dt \qquad (14)$$

Using Eq. (4)-(12), we get

$$C^T - 1^T + 2C^T.XY.S = 0 \qquad (15)$$

With choosing $N=14$, and solving (15), The estimated values of $y(x)$ are given in table1 and

The exact solution is $y(x)=e^{-x^2}$.

Example 2.

Consider the given IE

$$y(x) = x + \int_0^x (t-x)y(t)dt \qquad (16)$$

Using Eq. (4)-(12), we get

$$C^T - X^T - C^T.XY.S + C^T.H.(XY - P(0).X^T) = 0 \qquad (17)$$

With choosing $N=14$, and solving (17) are given.

C=[0,0.90350, 0, - 0.06304 ,0,0.00101,0,-7×10^{-6},0,2×10^{-8},0,-7×10^{-11},0,1×10^{-13}]

Table 1: Estimated and exact values of y(x)

x	Estimated	Exact
-1.0	0.36787	0.36787
-0.8	0.52729	0.52729
-0.6	0.69767	0.69767
-0.4	0.85214	0.85214
-0.2	0.96078	0.96078
+0.0	1.00000	1.00000
+0.2	0.96078	0.96078
+0.4	0.85214	0.85214
+0.6	0.69767	0.69767
+0.8	0.52729	0.52729
+1.0	0.36787	0.36787

The estimated solution is

Example 3.

Consider the given IE

$$-\frac{1}{12}x^2(y(x)+5) = \int_0^x (t-x)y(t)dt \qquad (18)$$

Using Eq. (4)-(12), we get

$$C^T.X^2Y + 5X^{2^T} + 12C^T.XY.S - 12C^T.H.(XY - P(0).X^T) = 0 \qquad (19)$$

With choosing $N=14$, and solving (19), The estimated and of $y(x)$,

C=$[\frac{4}{3},0,\frac{2}{3},0,0,0,0,0,0,0,0,0,0,0]^T$ and exact solution is $y(x)=x^2+1$.

Conclusion

The operational matrix of integral legendre function ,S, and other corresponding operators matrices and orthogonality of legendre functions, are used to solve IE. The present method reduces a IE into a set of algebraic equation and provide an exact solution almost. This method can be expanded for many IE.

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