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On the Approximative Solution of Boundary Value Problems by Collocation

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ABSTRACT

This paper concerns with the use of B-Splines to approximate a solution of a differential equations by collocation. The effect of knot placement on the accuracy of approximation is considered and numerical examples are given to illustrate the effectiveness of knot sequence.

Key words: Collocation methods, B-splines, knot sequence

INTRODUCTION

The numerical solution of boundary value problem is a topic in which active research is currently underway. There are number of methods used to solve boundary value problems. The most important of these probably the collocation method. For discussion referring to collocation method (Reddien, 1979) (Deuflhard, 1979) and (Ascher *et al.*, 1985). For an important collocation computer code, (Ascher *et al.*, 1981) and (Ascher *et al.*, 1981). In this study we use B-splines in the numerical solution of an initial boundary value problems by collocation. This method provides a strategy by which we can attack many problems in applied mathematics. Rayleigh Ritz method or Galerkin's method could be made quite effective if one were to give up on using polynomials or other analytic functions as trial function and used piecewise polynomial instead.

Numerical solution technique

Now let us consider how this method works (Boor *et al.*, 1973). We look for approximating a function g on [a,b], which is given to us implicitly, as a solution of the differential equation

$$(D^{m}g)(x) = F(x; g(x), K, (D^{m-1}g), \text{ for } x \in [a,b] \quad (2.1)$$

with boundary conditions

$$\sum_{j=1}^{m} w_j \, (D^{j-1}g)(x_i) = \beta_i g = c_i, \; i = 1, 2, K, m. \quad (2.2)$$

Where $F = F[X; Z_0, K, Z_{m-1}]$ is a real valued function on R^{m+1} and we will assume it to be sufficiently smooth and w_{ij} are constants and the points x_i satisfy $a \le x_1 \le K \le x_m \le b$. β_1, β_2, K, β_m are continuous linear functionals in $Ca^{(m-1)}$ and c_1, c_2, K, c_m are known constants. These boundary conditions are linear, the differential equations is nonlinear. Since we will linearize (2.1) in the computations, we could have made these conditions nonlinear as well (Wittenbrink, 1973).

Since (2.1) is nonlinear, (2.1-2.2) may have many solutions. Therefore we require that there be a neighborhood around the specific solution g and we will start our iterative process within this neighborhood in order to converge to this particular solution.

We intent to approximate g by piecewise polynomial (pp) functions using collocation. That is, we determine a pp function f so that it exactly satisfies the differential equations at certain points, the collocation points. We look $f \in P_{k+m,\xi} \cap C^{(m-1)}$ for which

$$(D^m f)(\tau_i) = F(\tau_i; f(\tau_i), K, (D^{m-1} f)(\tau_i)), \text{ for } i=1,K,kl \tag{2.3}$$

$$\beta_1 f = c_i, \; i = 1,2,K,m. \tag{2.4}$$

Here $P_{k,\xi}$ denotes the linear space of pp functions of order k with breakpoint sequence ξ.

We choose the collocation points per subintervals and distributed the same in each subinterval with. $-1 \le P < P_2 < K < P_k \le 1$. We calculate these points as follows;

$$\tau_{(i-1)k+v} = [\xi_{i+1} + \xi_i + P_v(\xi_{i+1} - \xi_i)]/2, v = 1,2,K\ k; i = 1,2,K,l.$$

We choose P as the zeros of the k-th Legendre polynomial. The reason for such a selection can be given with the following theorem (Boor *et al.*, 1978).

Theorem. Assume that the function F in (2.1) is sufficiently smooth in a neighborhood of the curve $[a, b] \to R^{m+1}: x \to (x; g(x), K, D^{m-1} g(x))$.

Assume further that the collocation points $P = (P_i)_{i=1}^k$ in $[-1,1]$ has been chosen such that $\int_{-1}^{1} q(x) \prod_{i=1}^{k} (x - p_i) dx = 0$ for every $q \in P_s$. Then the solution f near g of the approximate problem (2.3-2.4) satisfies

$$\|D^{\cdot} g - D^{\cdot} f\| \le \text{const} |\xi|^{\cdots}, i = 0, k, m \tag{2.5}$$

At the breakpoints, the approximation is of even higher order and satisfies

$$\left|D^i (g - f)(\xi_j)\right| \le \text{const} |\xi|^{k+s}, j = 1,1,K,l+1; i = 0,K,m-1. \tag{2.6}$$

Here const depends on F,g and k, but does not depend on ξ.

Since the problem (2.3-2.4) is nonlinear, in general, we need to use some iterative scheme for its solution. We can solve by Newton's method starting with a sufficiently close initial guess f_0, that is (2.3-2.4) has a solution

$$f = \lim_{r \to \infty} f_r$$

with $f_r + 1$ the solution $\in P_{k+m,\xi} \cap C^{(m-}$ of the linear problem.

$$(D^m y)(\tau_i) + \sum_{j<m} v_j(\tau_i)(D^j Y)(\tau_i) = h(\tau_i) \;\; i = 1,K,kl, \tag{2.7}$$

where

$$y_j(x) = -\left(\frac{\partial F}{\partial Z_j}(x; f_r(x), K, (D^{m-1} f_r)(x)\right), \; j = 0,K,m-1 \tag{2.8}$$

and

$$h(x) = F(x; f_r(x), K, (D^{m-1} f_r)(x)) + \sum_{j<m} v_j(x)(D^j f_r)(x) \tag{2.9}$$

The function y in (2.7-2.9) is a linear combination of appropriate B-splines. Let $t = (t_i)_1^{n+k+m}$ be the nondecreasing sequence which contains each of ξ_1 and ξ_{i+1} k+ m times and each interiorbreak point ξ_2, K, ξ_1, k times. Then n= kl+ m and

$$P_{k+m,\xi} \cap C^{(m-1)} = \$_{k+m,1}$$

Therefore the unknown function y can be written in the form

$$y = \sum_{(j=1)}^{n} \alpha_j B_{j,k+m,l}$$

We can determine y by determining its B coefficient vector α . This gives the linear system

$$\sum_{(j=1)}^{n} (B_j)(\tau_i)\alpha_j = h(\tau_i), i = 1, K, kl,$$

$$\sum_{(j=1)}^{n} (\beta_i B_j)\alpha_j = c_i, i = 1, K, m \qquad (2.10)$$

Where linear differential operator L is defined by $Ly = D^m y + \sum_{j<m} V_j(.) D^j y$

The following theorem gives sufficient condition for the existence of discrete solutions of boundary value problems.
Theorem. Let g(x), F (x; z_0, z_1, K, z_{m-1}) and $\left(\frac{\partial F}{\partial Z_k}\right)$, $(x; z_0, z_1, K, z_{m-1}$ be functions defined and continuous for

$$|z_k - g^{(k)}(x)| \le \delta, a \le x \le b \ (0 \le k \le m-1, \delta \ge 0)$$

Let 0 = *y* be only trivial solution of the homogeneous equation 0 = *(m) y* satisfying the boundary condition (2.2). If the linear homogeneous equation.

$$y^{(m)}(x) - \sum_{k=0}^{m-1} \frac{\partial f\,(x; g, g^{I}, K, g^{(m-1)})}{\partial Z_k} y^{(k)} = 0$$

has only trivial solution under the boundary condition (2.2), then there exist a number $\sigma > 0$ so that unique solution of the problem (2.1-2.2) can be found inside the sphere $\|w - g^{(m)}\| \le \sigma$

The Effect of knot placement on the accuracy of the Spline approximation

Construction of the piecewise polynomials depends on partition of the interval which is an important matter since every partition leads to a different approximation. It was suggested by (Boor *et al.*, 1978) that we place the breakpoints ξ_2, K, ξ_m so as to minimize

$$\max_i |\Delta\xi_i|^k \|D^k g\|_{(i)} \qquad (3.1)$$

For this purpose the following analysis should be considered .

$$s(\alpha, \beta) = |\beta - \alpha|^k \sup_{a<x<\beta} |(D^k g)(x)|$$

is a continuous function of á and â and monotone, increasing in â and decreasing in α when D^k g is continuous. In order to minimize (3.1) we choose ξ_2, K,ξ_1 so that

$$|\Delta\xi_i|^k \|D^k g\|_{(i)} = \text{constant for } i = 1, K, l. \qquad (3.2)$$

It is not easy task to find appropriate placement of ξ_i's since we don't know $D^k g$. Let us rewrite (3.2) as

$$\Delta\xi_i \| |D^k g|^{l/k} \|_{(i)} = \text{constant for } i = 1, K, l \qquad (3.3)$$

The last equality reduces the appropriate determining of ξ_2, K, ξ_l such that

$$\int_{\xi_i}^{\xi_{i+1}} |(D^k g)(x)|^{1/k}\, dx = \frac{1}{l}\int_a^b |(D^k g)|^{l/k} dx,\ i = 1, k,$$

This latter problem can be easily solved by replacing the function $|D^k g|$ by some piecewise constant function $h \approx |D^k g|$. Then

$$I(x) = \int_a^x (h(s))^{l/k}\, ds$$

is continuous and monotone increasing piecewise linear function. Hence its inverse I^{-1} is defined. It is required to evaluate the function I^{-1} at the l-1 points il (b)/l , l= 1, K, l-1.

Then we first determine a piecewise constant approximation h to the function $|D^k g|$. It makes no difference whether we construct the piecewise constant function h or the continuous piecewise linear function:

$$H(x) = \int_a^x (h(s))\, ds$$

It is possible to determine the function H(x) as an element belonging to $P_2, \xi \cap C$ by considering

$$\text{var}_{[a,x]} D^{k-1} f_\xi \approx \text{Var}_{[a,x]} D^{k-1} = \int_a^x |(D^k g)(s)| ds$$

We choose $h \in P_1, \xi$ such that

$$h(x) = \begin{pmatrix} 2\dfrac{|\Delta f_{i-1/2}|}{\xi_3 - \xi_1} & \text{on } [\xi_1 - \xi_2] \\ \dfrac{|\Delta f_{i-1/2}|}{\xi_{i+1} - \xi_{i-1}} + \dfrac{|\Delta f_{i-1/2}|}{\xi_{i+2} - \xi_1} & \text{on } [\xi_i, \xi_{l+1}], i = 2, K, l-1, \\ \dfrac{|\Delta f_{i-1/2}|}{\xi_{i+1} - \xi_{l-1}} & \text{on } [\xi_1 - \xi_{i+1}] \end{pmatrix}$$

where we have used the abbreviation $f_{i+1/2} = D_{k-1} f_\xi$ on $[\xi_i, \xi_{l+1}]$, all it

If we sum up the process;

i) Choose the breakpoints ξ_i's and an initial solution f_0 .

ii) Obtain f_ξ by using Newton's method.

iii) For better approximation, obtain $h \in P_1, \in_\xi$ with the use of f_ξ .

iv) Determine the number *l* and the breakpoints $\xi_{i+1} = I^{-1}(il(b)/l)$, l= 1, K, l-1

v) Replace f_0 by f_1 and repeat the process.

The method discussed above have been applied to the following problems and the results obtained are given below.

Numerical results

In this section the method discussed above were tested on two problems.

Example. $0.005y'' + y^2 = 1,\ 0 \le x \le 1$

with the following boundary condition:
$y'(0) = y(1) = 0$

If we linearize the problem about the point $y= y_0$ by Newton's method we obtain

$0.005\, y'' + 2y_0 y = 1 + y^2_0$
$y'(0) = y(1) = 0$

Let $y_0 = x^2 - 1$ be initial solution.
Let $f \in P_{5+2} \cap C^{(1)}$. We subdivide the interval [1 , 0] into five subintervals and select the following points initially;
0.00 0.25 0.5 0.75 1.

In each iteration we have used the most recent approximation to the solution as the current
guess f_r together with a different knot sequence, which is obtained via f_r .
The knot sequences obtained are shown as below.

0.00	0.41444	0.63868	0.82429	1.00
0.00	0.44955	0.69137	0.85175	1.00
0.00	0.44042	0.70181	0.85773	1.00
0.00	0.44157	0.70404	0.85902	1.00
0.00	0.44115	0.70454	0.85931	1.00
0.00	0.44119	0.70464	0.85937	1.00
0.00	0.44117	0.70467	0.85939	1.00

The best results are obtained using the last knot sequence. The solution changes rapidly in the interval [0.75,1] . Therefore numerical results obtained are given for this interval.

Point	Exact value	Approx. value
0.760	-0.9900401	-0.9900425
0.775	-0.9865633	-0.9865644
0.790	-0.9818766	-0.9818751
0.820	-0.9670590	-0.9670580
0.835	-0.9556199	-0.9556212
0.850	-0.9402484	-0.9402495
0.865	-0.9196254	-0.9196376
0.880	-0.8920158	-0.8920472
0.910	-0.8061453	-0.8060719
0.925	-0.7413051	-0.7412860
0.940	-0.6561230	-0.6561938
0.955	-0.5452587	-0.5453423
0.970	-0.4027639	-0.4027616
0.985	-0.2226500	-0.2225955
1.000	-0.0000000	-0.0000000

Example: $y'' = e^y,\ 0 \le x \le 1$

with the following conditions

$y(0) = y(1) = 0$

If we linearize the problem about the point $y= y_0$ by Newton's method we obtain

$y'' - e^{y0}\, y = (1 - y_0) e^{y0}$
$y'(0) = y(1) = 0$

Let $y_0 = x^2 - x$ be initial solution.

Let $f \in P_{4+2} \cap C^{(1)}$ The interval [1, 0] is divided into five subintervals and we choose the following points initially;

0.00 0.20 0.40 0.60 0.80 1.00

The knot sequences obtained are as follows

0.00	0.19640	0.39710	0.60290	0.80360	1.00
0.00	0.19727	0.39787	0.60213	0.80273	1.00
0.00	0.19712	0.39753	0.60200	0.80271	1.00
0.00	0.19707	0.39741	0.60205	0.80279	1.00
0.00	0.19709	0.39747	0.60207	0.80275	1.00

The following results are found for the last knot sequence given above.

Point	Exact value	Approx. value
0.00	-0.0000003	0.0000000
0.08	-0.0339285	-0.0339321
0.16	-0.0616664	-0.0616678
0.24	-0.0833838	-0.0833853
0.32	-0.0992101	-0.0992133
0.40	-0.1092380	-0.1092379
0.48	-0.1135254	-0.1135287
0.56	-0.1120970	-0.1120984
0.64	-0.1049445	-0.1049457
0.72	-0.0920267	-0.0920300
0.80	-0.0732687	-0.0732685
0.88	-0.0485596	-0.0485632
0.96	-0.0177504	-0.0177519

Better results can be obtained by increasing the order of polynomials and acccuracy in the iterative process. But the most effective method beyond these is the repositioning of the breakpoints. As a different approximation to the solution we change the place of the knots in each iteration and we observed that accuracy is increased and the number of iteration is reduced.

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