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Anisotropy in the Ray Tracing with the Emphasis on Hexagonal Symmetry (TIV)

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Abstract: For ray tracing purposes, the difference between the phase and group velocities is clarified in order to derive numerically the change in ray velocity due to anisotropy. The simplest anisotropy case of broad geophysical applicability is the ***transverse isotropy*** or ***hexagonal symmetry***. The main notations introduced by Auld, 1990 to describe the ***transverse isotropy*** of vertical symmetry axis (TIV) have been used.

Key words: Anisotropy phase velocity group velocity transverse isotropy

Introduction

Seismic anisotropy is the variation of velocity as a function of the signal propagation direction through a medium. The important features of wave propagation in anisotropic solids are I) the variation of wave velocities with direction ii) the three-dimensional (3-D) displacement of the particle, which leads to shear-wave splitting and iii) the propagation of energy deviated both in velocity and direction from the direction of the phase propagation. A complete theory and the main notations describing the transverse isotropy of vertical symmetry axis (TIV) have been introduced following Auld, 1990 which are used to drive Thomsen's formulas'.For ray tracing purposes it is important to clarify the differences between the phase and group velocities in order to numerically derive the change in ray velocity due to anisotropy. Group velocity is the speed at which wave energy travels in a given direction radially outward from a point source in a homogeneous elastic anisotropic medium (Winterstein, 1990). Phase velocity is the velocity in the direction of the phase propagation vector, normal to the surface of constant phase (Crampin, 1989).The simplest anisotropy case of broad geophysical applicability is the t***ransverse isotropy*** with a vertical symmetry axis (TIV). It serves as a good introduction to anisotropy for geophysicist and helps to define the basic terminology and methodology for anisotropy studies (Alkhalifah and Tsvankin, 1995).

Equations of motion

The equation of motion is obtained using Newton's law

$$\int_{\Omega} \vec{\nabla} T \, dv = \int_{\Omega} \rho \frac{\partial \vec{\upsilon}}{\partial t} dV \Rightarrow \vec{\nabla} T (\vec{x}, t) = \rho \frac{\partial \vec{\upsilon}}{\partial t} (\vec{x}, t) \tag{1}$$

ρ is the density of the medium and $\vec{\upsilon}(\vec{x}, t) = \dfrac{\partial \vec{u} (\vec{x}, t)}{\partial \vec{t}}$ is the particle velocity. Using equation (1), straightforwardly have

$$\nabla \frac{\partial T}{\partial t} = \rho \frac{\partial^2}{\partial t^2} \quad (2)$$

A uniform plane wave $\upsilon(x, t)$ propagating along the $\vec{l}(l_1 = l_x, l_2 = l_y, l_3 = l_z)$ direction is proportional to $e^{i(\omega t - \vec{k}.\vec{x})}$ where $\| l \| = 1$ and k is the wavenumber; is the velocity of advance of wavefront (constant time i.e. constant phase) and is called the phase velocity. Using $\upsilon \equiv e^{i(\omega t - \vec{k}.\vec{x})}$ and the relations from operators act on a plane wave substituted in the wave equation for general homogeneous media, gives the dispersion relation

$$\sum_{j=1}^{3} \frac{\tau_j}{\rho} \upsilon j = \left(\frac{\omega}{K}\right)^2 \upsilon_i, \; i=1,2,3 \quad (3)$$

where τ a 3×3 matrix called the Christoffel matrix: its elements are functions only of the plane wave propagation direction l and of the stiffness constants c_{KL} of the medium. The dispersion relation (3) is an eigenvalues problem

($\left(\frac{\omega}{K}\right)^2$ are the eigenvalues). It has the unique solution $\upsilon = 0$ (i.e. there is no propagation in the medium) if the determinant of the system is non-zero; since this is not physically satisfactory we want a zero determinant

$$\left[\frac{\tau}{\rho} - \left(\frac{\omega}{K}\right)^2 I\right] = 0, \quad (4)$$

where, I is identity matrix.

Solving equation (4) gives 3 possible expressions for the phase velocity $\frac{\omega}{K}$, only waves having one of these phase velocities propagate in the medium.

We can calculate an eigenvector associated to each eigenvalues that corresponds to the polarization of the wave propagating with the phase velocity. Mathematically the three eigenvectors are mutually orthogonal which means that physically the three polarizations are in the direction of propagation that is the quasi-longitudinal wave and is simply denoted P. Another eigenvector is orthogonal to the first one but not to the direction of propagation that is the quasi-traverse shear wave and is denoted SV. The last eigenvector is orthogonal to the direction of propagation (and to the other eigenvectors) that is the exactly traverse shear wave and is denoted SH.

Hexagonal symmetry and vertical axis

This system has one of the axes of symmetry such that a rotation by an arbitrary angle around this axis does not change the tensor i.e. the tensor behaves isotropically in the plane perpendicular to this axis. Because of this, the symmetry is also sometimes called ***transverse***

isotropy, especially in the case when the axis of rotational symmetry coincides with the -axis of the coordinate system.

A crystal with a hexagonal symmetry has a lattice (of atoms) built from two equal vectors in the horizontal plane, with an angle of 120 degrees and a third vector orthogonal to the horizontal plane. This kind of lattice allows a great number of symmetries such that the appearance of the lattice after the transformation remains unchanged. Thus, stiffness constants are not all independent (i.e. there are less than 21 constants to describe such crystal) since constraints are imposed on them by the principle that symmetrically equivalent directions in a crystal must have equivalent elastic properties (Auld, 1990).

In the case of the hexagonal symmetry, the equation of motion (4) is give by

$$k^2\begin{bmatrix} c_{11}l_x^2+c_{66}l_y^2+c_{44}l_z^2 & (c_{11}-c_{66})l_x l_Y & (c_{13}+c_{44})l_x l_z \\ (c_{11}-c_{66})l_x l_Y & c_{66}l_x^2+c_{11}l_y^2+c_{44}l_z^2 & (c_{13}+c_{44})l_y l_z \\ (c_{13}+c_{44})l_x l_z & (c_{13}+c_{44})l_y l_z & c_{44}+(l_x^2+l_y^2)+c_{33}l_z^2 \end{bmatrix}\begin{bmatrix}\upsilon_1\\ \upsilon_2\\ \upsilon_3\end{bmatrix}=\rho\omega^2\begin{bmatrix}\upsilon_1\\ \upsilon_2\\ \upsilon_3\end{bmatrix} \tag{5}$$

Thus there are only five independent elastic constants for describing a crystal with hexagonal symmetry: c_{11}, c_{13}, c_{33}, c_{44}, c_{66}.

Equation (1) is simplified by imposing the propagation in the (x,y) plane; since the Christoffel equation can be shown to be symmetric with respect to an arbitrary rotation about z-axis, we could have chosen any meridian plane. We write the propagation vecto $\vec{l}$ as

$$\vec{l}(l_x=\sin\theta, l_y=0, l_z=\cos\theta) \tag{6}$$

where θ is the angle between the propagation direction and the vertical axis.
Solving the zero-determinant equation (4) yields the solutions satisfying

$$\left(\left(\frac{\omega}{K}\right)_1, \left(\frac{\omega}{K}\right)_2, \left(\frac{\omega}{K}\right)_3\right) = (\upsilon_p, \upsilon_{sv}, \upsilon_{sh}) \text{ satisfying}$$

$$\upsilon_p^2(\theta)=\frac{c_{11}\sin^2+c_{33}\cos^2+c_{44}+\sqrt{\left[\left(c_{11}\,c_{44}\right)\sin^2+\left(c_{44}\,c_{33}\right)\cos^2\right]+\left(c_{13}+c_{44}\right)^2\sin^2 2}}{2a}$$

$$\upsilon_{sv}^2(\theta)=\frac{c_{11}\sin^2+c_{33}\cos^2+c_{44}+\sqrt{\left[\left(c_{11}\,c_{44}\right)\sin^2+\left(c_{44}\,c_{33}\right)\cos^2\right]+\left(c_{13}+c_{44}\right)^2\sin^2\theta 2}}{2\rho}$$

$$\upsilon_{SH}^2(\theta)=\frac{c_{66}\sin^2\theta+c_{44}\cos^2\theta}{\rho} \tag{7}$$

Thomsen's notations

As introduced by Thomsen (1986), it is useful to recast equations (7) using notations involving only two elastic moduli (e.g. vertical P and S velocities) plus three measures of anisotropy. These three anisotropy coefficients should be nondimensional so that we can speak of a percentile of anisotropy, should be efficient combinations of elastic moduli (c_{11}, c_{13}, c_{33}, c_{44}, c_{66}) in order to simplify equations (7) and should reduce to zero in the case of isotropy. Some suitable combinations are $\varepsilon = \dfrac{c_{11} - c_{33}}{2c_{33}}, \gamma = \dfrac{c_{66} - c_{44}}{2c_{44}}$

$$\delta = \frac{1}{2} c_{33}^{2} [2(c_{13} + c_{44})^{2} - (c_{33} + c_{44})(c_{11} + c_{33} - 2c_{4}) \tag{8}$$

using the vertical P and S velocities

$$\alpha_0 = v_p(0) = \sqrt{\frac{c_{33}}{\rho}} \quad , \quad \beta_0 = v_{s,v}(0) = v_{sh}(0) = \sqrt{\frac{c_{44}}{\rho}} \tag{9}$$

Then equation (7) rewrites as

$$v_p(\theta) = \alpha_0 \sqrt{\varepsilon \sin^2\theta + D(\theta)} = \alpha_0 \alpha_p(0)$$

$$v_{sv}(\theta)\, \beta_0 \sqrt{1 + \frac{\alpha_0^2}{\beta_0^2} \epsilon \sin^2\theta - \frac{\alpha_0^2}{\beta_0^2} D(\theta)} = \beta_0 a_{sv}(\theta) \tag{10}$$

$$V_{SH}(\theta) = \beta_0 \sqrt{1 + 2\gamma \sin^2\theta} = \beta_0 a_{sh}(\theta)$$

Where is given by

$$D(\theta)\, \frac{1}{2} \sqrt{q^2 + 4\delta \sin\theta + 4\sin^4\theta[\epsilon(q + \epsilon) - \delta]} - \frac{q}{2} \tag{11}$$

with

$$q = 1 - \frac{\beta_0^2}{\alpha_0^2} = \frac{c_{33} - c_{44}}{c_{33}} \tag{12}$$

where ε, δ and γ are called Thomsen's parameters and are convenient variables to support calculus; where he derived velocity formulas for weak values of these parameters. However, inputs to describe anisotropy in some topics include the five independent elastic constants c_{11}, c_{13}, c_{33}, c_{44} and c_{66} (from which Thomsen's parameters are evaluated).

Phase velocity and group velocity

In a general homogeneous elastic medium, where the velocity is constant in any given direction, it is obvious that a particle moves along a straight line; the energy propagates along this line with the group velocity and the direction of propagation makes the group angle φ with the vertical axis. This line defines a ray.

Because of anisotropy, the wavefront, that is the positions of constant propagation time, is non spherical. The wave vector $\vec{kl}$ is locally perpendicular to the wavefront and makes the phase angle with the vertical axis. The phase velocity that measures the velocity of advance of the wavefront along the direction $\vec{kl}$ is given by the Thomsen's formula (10).

Fig. 1 shows that a point on the wavefront (fixed propagation time) can be reached either traveling either with the group V (φ) velocity along the direction making the group angle φ with the vertical axis, or traveling with the phase velocity along the direction that is perpendicular to the wavefront and that makes the phase angle with the vertical axis. Note that the phase velocity direction does not start at the source point.

Thud, in a homogeneous medium, wavefronts are defined by

$$t(x,z) = \frac{\sqrt{x^2+z^2}}{\upsilon(\theta)} = \frac{\sqrt{x^2+z^2}}{V(\theta)} = \text{Cosntant} \tag{13}$$

from

$$t(x,z) = \frac{\sqrt{x^2+z^2}}{V(\phi)} = \frac{\sqrt{x^2+z^2}}{V(\arctan(\frac{x}{z}))} \tag{14}$$

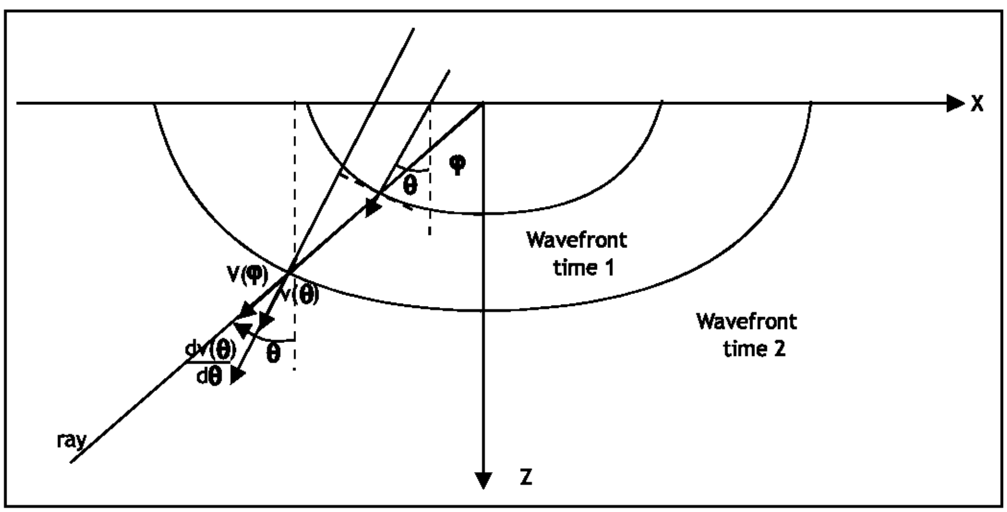

Fig. 1: phase (wavefront) angle at two consecutive times and group (ray) angle

We see that

$$t(\lambda x, \lambda z) = \lambda t(x, z) \quad . \quad \forall > 0. \tag{15}$$

In other words, the traveltime along the ray is a linear function of the source-wavefront distance. Also, along a ray, the direction of the normal vector to the wavefront $\vec{l} = (\frac{-\partial t}{\partial z}, \frac{-\partial t}{\partial x})$ is constant. Thus, to a phase angle corresponds one and only one group angle definingφ a ray, whatever the propagation time.

The phase velocity is defined as the projection of the group velocity on the wavefront normal direction (Fig. 1). Thus the group velocity is given in terms of the phase velocity by

$$V^2[\varphi(\theta)] = \upsilon^2(\theta) + \left(\frac{d\upsilon}{d\theta}\right)^2 \Rightarrow V[\phi(0)] = \upsilon(\theta)\sqrt{1 + \frac{1}{\upsilon^2(\theta)} \ 1\left(\frac{d\upsilon}{d\theta}\right)^2} \tag{16}$$

and using the general form of equation (10) we get

$$V[\varphi(\theta)] = \upsilon(\theta)\, a(\theta) + \sqrt{1 + \frac{1}{\upsilon^2(\theta)}\left(\frac{d\upsilon}{d\theta}\right)^2} = \upsilon(0)\, A(\phi) \tag{17}$$

The general relation between group angle and phase angle is (Fig. 1)

$$\tan[\phi - (\theta)] = \frac{\frac{d\upsilon}{d\theta}}{\upsilon(\theta)} \Rightarrow \tan\phi = \frac{\tan\theta + \frac{1}{\upsilon}\frac{d\upsilon}{d\theta}}{1 - \frac{\tan\theta}{\upsilon}\frac{d\upsilon}{d\theta}} \tag{18}$$

In anisotropic media, wavefronts traveling outward from a point source are not, in general, spherical as a result of dependence of velocity upon direction of propagation. Shown in Fig. 1 are two wavefronts in space that are separated by unit time. The group velocity, v(φ), denotes the velocity with which energy travels from the source, while the phase velocity, , is the velocity with which a wavefront propagates at a local point, that is, the propagation velocity of the parallel plane-wave component. Here ϕ is the group angle, also called group angle and specifies the direction of the ray from the source point to the point of interest. And is the ϕ phase angle, also called wavefront-normal angles, it specifies the direction of the vector that is normal to the wavefront, in general, different from group angle at any point of propagation, except at certain singular points.

This research presents the anisotropy with some details in the transverse isotropy with vertical axis of symmetry. The group velocity variation as function of the group angle needs is derived from the phase velocity variation with the phase angle. This has been achieved using simplifications of notations introduced by Auld, 1990. ***Transverse isotropy*** (TIV) is described by five elastic parameters c_{11}, c_{13}, c_{33}, c_{44}, c_6.

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