Pakistan Journal of Applied Sciences 3(2): 71-75, 2003

Principal Ideal Rings

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Abstract: In this paper we continue to extend ring concepts. Here we define principal ideal rings for commutative rings (not necessarily with identity) and prove that this definition is equivalent to the usual definition in the case of a ring with identity. Then we generalize some results for principal ideal rings. We study direct sums, direct summands and quotient rings. We show that every Euclidean ring is a principal ideal ring.

Key Words: Euclidean Rings, Euclidean Algorithms, Principal Ideal, Principal Ideal Ring

Introduction

In 1949 Motzkin had proved a necessary and sufficient condition for an integral domain to be a Euclidean ring. Two generalizations of this concept came by Fletcher in 1969 and by Samuel in 1971. In 1977, the first example of a Euclidean domain without any integer-valued algorithm was given by Hiblot. In 1978-1987, Nagata, gave his definitions. We generalize the concept of Euclidean rings to commutative rings which do not necessarily have an identity (Agargun, 1997). We give seven new definitions and the relations among them. Our definitions are equivalent to the usual definitions in the case of a ring with identity. We generalize Motzkin's work to give a useful result for finding the Euclidean algorithm (smallest algorithm) for commutative rings. Also some examples of Euclidean rings and the smallest algorithms are given in this work. (Agargun, 1997). In generalising the concept of a Euclidean ring to commutative rings we take the opportunity of examining the connections between the many different definitions. Hence overall we have the three types of Euclidean ring. We generalize some results using these group definitions (Agargun, 2000). We study direct sums, direct summands, quotient rings and rings of fractions. We also consider the Euclidean property with respect to a partially ordered set with minimum condition instead of a well-ordered set.

Let R and R' be commutative rings, R' with identity and let $\theta : R \to R'$ be a monomorphism where $\theta(R)$ is an ideal of R'. Let us call this kind of monomorphism as a G-monomorphism. Suppose a, b $\in$R, W is a well-ordered set and φ is a mapping into W. The three areas for choice are

i. $\varphi : R\backslash\{0\} \to W$ or $\varphi : R \to W$
ii. b any element of R or $b \neq 0$,
iii. $r = 0$ or $\varphi(r) < \varphi(b)$, or
$r = b$ or $\varphi(r) < \varphi(b)$, or
$\varphi(r) < \varphi(b)$.

Therefore we have seven possible definitions of Euclidean rings.

Definition 1.1 A map $\varphi : R\backslash\{0\} \to W$ is called a Euclidean algorithm (with respect to $\theta : R \to R'$) if for all a, b $\in$R, $b\neq 0$, there exist $q' \in R'$ and $r \in R$ such that $\theta(a) = q'\,\theta(b) + \theta(r)$ where $r = 0$ or $\varphi(r) < \varphi(b)$.

Similarly, we have the other definitions by consideration the three areas for choice.

Definition 1.2: $\varphi : R\backslash\{0\} \to W$, $b\neq 0$, $r = b$ or $\varphi(r) < \varphi(b)$.

Definition 1.3: $\varphi : R \to W$, $b\neq 0$, $r = 0$ or $\varphi(r) < \varphi(b)$.

Definition 1.4: $\varphi : R \to W$, $b\neq 0$, $r = b$ or $\varphi(r) < \varphi(b)$.

Definition 1.5: $\varphi : R \to W$, $b\neq 0$, $\varphi(r) < \varphi(b)$.

Definition 1.6: $\varphi : R \to W$, any b, $r = 0$ or $\varphi(r) < \varphi(b)$.

Definition 1.7: $\varphi : R \to W$, any b, $r = b$ or $\varphi(r) < \varphi(b)$.

Here, we note that if R has an identity then by taking θ to be the identity homomorphism then our definitions are the same as the definitions from Amono (1985); Lenstra (1974); Kanemitsu and Yoshida (1986); Nagata (1985 and 1987) and Samuel (1971).

Let us mean by the word "equivalent" that if φ is an Euclidean algorithm according to one definition then it will be a Euclidean algorithm in the sense of the other. Then the connections between these new definitions are the same with the results of our paper on Euclidean rings (Agargun and Fletcher, 1995). Here we only need to consider the elements of R with θ according to the definitions. By allowing W and φ to chance all these definitions become equivalent.

Definition 1.8: Let R be a commutative ring. If there exists a commutative ring with identity R', a G-monomorphism $\theta : R \to R'$, a well-ordered set W, an algorithm $\varphi : R \to W$ satisfying one of Definitions 1.1-1.7, say Definition 1.6, then we say that R is a Euclidean ring with respect to $\theta : R \to R'$ and $\varphi : R \to W$.

We consider now the special case where the well-ordered set is taken to be $Z^{+}\cup\{0\}$. In general Definition 1.5 does not imply Definition 1.6 and Definition 1.3 does not imply Definition 1.6 since the well-ordered set cannot change. However Definition 1.3 does imply Definition 1.5 since $Z^{+}\cup\{0\}$ does not have maximal element. Therefore, the case of $Z^{+}\cup\{0\}$ splits up in the two cases corresponding to Definitions 1.1-1.5 and Definitions 1.6-1.7.

Hence overall we have three types of Euclidean ring as:
i. Definition Group A: Euclidean rings with respect to a general well-ordered set using Definitions 1.1-1.7.
ii. Definition Group B: Euclidean ringss with respect to $Z^+ \cup \{0\}$ using Definitions 1.1-1.5.
iii. Definition Group C: Euclidean rings with respect to $Z^+ \cup \{0\}$ using Definitions 1.6-1.7.

Principal Ideal Rings: In this section,we start by giving a well-known definition.

Definition 2.1: Let R be a c.r.w.1. An ideal I of R is called a principal ideal of R if I is generated by a single element $a \in R$ that is $I = Ra = \{ra : r \in R\}$. This is denoted by "(a)".

Here we also give the following definition of a principal ideal with respect to a G-monomorphism. For the identity G-monomorphisms it is equivalent to the previous definition.

Definition 2.2: Suppose R is a commutative ring and $\theta: R \to R'$ is a G-monomorphism. Then an ideal I of R is called a principal ideal (wrt. $\theta : R \to R'$) if $\theta(I)$ is a principal ideal of R'.

We now give our definition of Principal Ideal Ring (PIR) for commutative rings.

Definition 2.3: Let R be a commutative ring, R' a c.r.w.1 and $\theta : R \to R'$ a G-monomorphism. Then R is called a Principal Ideal Ring (PIR) (wrt. $\theta : R \to R'$) if every ideal I' of R' such that $I' = \theta(I)$ for some ideal I in R is a principal ideal of R'. In other words R is a PIR (wrt. $\theta : R \to R'$) if every ideal I of R such that $\theta(I)$ is an ideal of R' is principal, i.e. I is a principal ideal (wrt. $\theta : R \to R'$).

We know the definition of PIR from the text books in the cases when R has an identity and when it does not. In both cases our definition is equivalent to the usual definitions subject to a given G-monomorphism. We show these connections in Theorems 2.5 and 2.6. First let us give some examples.

Example 2.4:

i. nZ is a PIR wrt inc.: $nZ \to Z$ since every ideal of Z is a principal ideal.
ii. nZ is a PIR wrt. $\theta : nZ \to nZ \times Z$ where $\theta(nk) = (nk, 0)$. For suppose $A \times \{0\} = \theta(A)$ is an ideal of $nZ \times Z$. Since A is an ideal of nZ, it is an ideal of Z. Therefore A is principal ideal of Z. We take A = (a) in Z for $a \in A$. Then $A \times \{0\} = ((a, 0))$. Therefore $\theta(A)$ is a principal ideal of $nZ \times Z$ and nZ is a PIR (wrt. $\theta : nZ \to nZ \times Z$).

The following theorem is immediate from Definitions 2.1-2.3.

Theorem 2.5: Suppose R is a c.r.w.1. Then R is a PIR (wrt. $1 : R \to R$) if and only if every ideal of R is principal.

Let us consider the G-monomorphism $\theta : R \to R \times Z$ given by $\theta(r) = (r, 0)$. Here we have the following result. This is the usual definition of a principal ideal in a commutative ring (Hungerford, 1974) and according to our definition it is a special case.

Theorem 2.6: Suppose R is a commutative ring and I is an ideal of R then I is a principal ideal (wrt. $\theta : R \to R \times Z$) if and only if there exists $a \in I$ such that $I = \{ra + na : r \in R, n \in Z\}$.

Proof: Suppose I is a principal ideal (wrt. $\theta : R \to R \times Z$). Then $\theta(I) = I \times \{0\}$ is a principal ideal in $R \times Z$. Therefore there exists $(a, 0) \in \theta(I)$ such that $((a, 0)) = \theta(I)$. Hence
$\theta(I) = \{(a,0)(r, n) : (r, n) \in R \times Z\} = \{(ra + na, 0) : r \in R, n \in Z\}$
and this implies $I = \{ra + na : r \in R, n \in Z\}$.
Conversely, suppose there exists $a \in I$ such that $I = \{ra + na : r \in R, n \in Z\}$. Hence
$\theta(I) = I \times \{0\} = \{(ra + na, 0) : r \in R, n \in Z\} = \{(a,0)(r, n) : (r, n) \in R \times Z\} = ((a, 0))$.
Therefore $\theta(I)$ is a principal ideal of $R \times Z$ and so I is a principal ideal (wrt. $\theta : R \to R \times Z$).

In Theorem 2.6, if R has an identity then $I = \{ra+na : r \in R, n \in Z\} = \{ra : r \in R\} = (a)$. Hence the following result is immediate.

Corollary: Suppose R is a c.r.w.1. Then R is a PIR (wrt. $\theta : R \to R \times Z$) if and only if R is a PIR (wrt. $1 : R \to R$).

Proposition 2.7: Suppose R is a commutative ring and S is a multiplicatively closed subset (m.c.set) of R. Then $\theta_S : R \to R_S$ defined by $\theta_S(r) = rs / s$ is a ring homomorphism but not a G-monomorphism in general.

Proof: Suppose $r_1, r_2 \in R$ then we have for $s \in S$,
$\theta_S(r_1 + r_2) = (r_1 + r_2)s / s = r_1s + r_2s / s = r_1s / s + r_2s / s = \theta_S(r_1) + \theta_S(r_2)$,
$\theta_S(r_1r_2) = (r_1r_2)s / s = r_1r_2ss / ss = r_1s / s \,.\, r_2s / s = \theta_S(r_1)\theta_S(r_2)$.
Therefore θ_S is a homomorphism. If S contains a zero divisor then clearly θ_S is not a monomorphism. Because, if rs = 0 for $0 \neq r \in R$ and $0 \neq s \in S$, then $\theta_S(r) = rs / s = 0 / s$.

Corollary: Suppose R is a commutative ring and S is a m.c.set of R which does not contain a zero-divisor. Then $\theta_S : R \to R_S$ defined by $\theta_S(r) = rs / s$ is a ring monomorphism but not a G-monomorphism in general.

Proof: From the previous proposition θ_S is a homomorphism. If rs / s = 0, then for $t \in S$ rst = 0 and this implies r = 0. Therefore θ_S is a monomorphism. But $\theta_S(R)$ is not an ideal of R_S in general. For we give a counter-example in which take R = 2Z and $S = 4Z \setminus \{0\}$. Therefore we write $\theta(2)2 / 4 = 2(4s) / (4s) \,.\, 2 / 4 \notin \theta_S(R)$.

Suppose $1: R \to R$ denotes the identity homomorphism. We consider a subset of R_S in Proposition 2.7, $< \theta_S(R), 1_{R_S} > = \{rs + ns / s : r \in R, s \in S, n \in Z\}$. Then we have the following proposition.

Proposition 2.8: Suppose R is a commutative ring and S is a m.c.set of R which does not contain a zero divisor. Then $\theta_S : R \to < \theta_S(R), 1_{R_S} >$ defined by $\theta_S(r) = rs / s$ is a G-monomorphism.

Proof: From Proposition 2.7 Corollary θ_S is a monomorphism. Suppose $r \in R$,
$r_1s + ns / s \in < \theta_S(R), 1_{R_S} >$. Then

$\theta_S(r)$ $(r_1s + ns / s) = rs_1 / s_1 \cdot (r_1s + ns / s) = (rr_1 + nr)s_1s / s_1s \in \theta_S(R)$.

Therefore $\theta_S : R \rightarrow < \theta_S(R), 1_{R_S} >$ is a G-monomorphism.

From Proposition 2.8, if R is a commutative ring and S is a m.c.set of R which does not contain a zero-divisor then

$\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}> = \{rs + ns / s : r \in R, s \in S, n \in Z^+\}$

defined by $\theta_S(r) = rs/s$ is a G-monomorphism. Therefore we have an equivalent result of Theorem 2.6 for θ_S :0 $R \rightarrow <\theta_S(R), 1_{R_S}>$.

Theorem 2.9: Suppose R is a commutative ring, S is a m.c.set of R which does not contain a zero-divisor and I is an ideal of R then I is a principal ideal (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$) if and only if there exists a $\in$I such that $I = \{ra + na : r \in R, n \in Z\}$.

Proof: Let I be a principal ideal (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$). Then $\theta(I)$ is a principal ideal in $<\theta_S(R), 1_{R_S}>$. Therefore there exists $as/s \in \theta(I)$ such that $(as/s) = \theta(I)$. Hence

$\theta(I) = \{(as/s\ (rt+nt/t) : rt+nt/t \in <\theta_S(R), 1_{R_S}>\} = \{ras + nas /s : r \in R, s \in S, n \in Z\}$ and this implies $I = \{ra + na : r \in R, n \in Z\}$.

Conversely, suppose there exists a $\in$I such that $I = \{ra + na : r \in R, n \in Z\}$. Hence

$\theta(I) = \{ras + nas /s : r \in R, s \in S, n \in Z\} = \{(as/s\ (rt+nt/t) : rt+nt/t \in <\theta_S(R), 1_{R_S}>\} = (as/s)$.

Therefore $\theta(I)$ is a principal ideal of $<\theta_S(R), 1_{R_S}>$ and hence I is a principal ideal (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$).

Therefore the following result is immediate.

Corollary: Suppose R is a commutative ring, S is a m.c.set of R which does not contain a zero-divisor and I is an ideal of R then I is a principal ideal (wrt. $\theta : R \rightarrow R \times Z$) if and only if I is a principal ideal (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$). Therefore we have that R is a PIR (wrt. $\theta : R \rightarrow R \times Z$) if and only if R is a PIR (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$).

If our G-monomorphism is into a PIR then we have immediately the following theorem.

Theorem 2.10: Suppose R is a commutative ring and R' a c.r.w.1. If R' is a PIR (wrt. $1 : R' \rightarrow R'$) and $\theta : R \rightarrow R'$ is a G-monomorphism, then R is a PIR (wrt. $\theta : R \rightarrow R'$).

Here we point out that in general R is a PIR (wrt. $\theta : R \rightarrow R'$) does not imply R is a PIR with respect to the restriction map $\theta : R \rightarrow <\theta(R), 1_{R'}>$. For a counter-example, we have that

$O_2 \oplus O_2$ is a PIR (wrt. $\theta : O_2 \oplus O_2 \rightarrow Z_4 \oplus Z_4$) (**Example 2.15**.(i)), but it is not a PIR (wrt. $\theta : O_2 \oplus O_2 \rightarrow <\theta(O_2 \oplus O_2), (1, 1)>$) since $\theta(O_2 \oplus O_2)$ is not a principal ideal of $<\theta(O_2 \oplus O_2), (1, 1)>$.

Theorem 2.11: Suppose R is a c.r.w.1 and S is a m.c.set of R which does not contain a zero-divisor. Then R is a PIR (wrt. $1 : R \rightarrow R$) if and only if R is a PIR (wrt. $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$).

Proof: Since R has a identity $<\theta_S(R), 1_{R_S}> = \theta_S(R)$ and hence $\theta_S : R \rightarrow <\theta_S(R), 1_{R_S}>$ becomes an isomorphism. Therefore the proof is immediate.

For ring of factions we have the following result.

Theorem 2.12: Suppose R is PIR (wrt. $\theta : R \rightarrow R'$) and S is a m.c.set of R not containing 0. Then R_S is a PIR (wrt. $\theta' : R_S \rightarrow R'_{\theta(S)}$ given by $\theta'(r/s) = \theta(r)/\theta(s)$).

Proof: We know from Theorem 3.2 of (Agargun, 2000) θ' is a G-monomorphism. Let $J_S = \theta(I_S)$ be an ideal of $R'_{\theta(S)}$, where $I_S = \{a/s : \theta(a)/\theta(s) \in J_S\}$ is an ideal of R_S. Consider $I = \{a \in R : as/s \in I_S\}$. Here we can easily see that I and $\theta(I) = \{\theta(a) \in \theta(R) : as/s \in I_S$, i.e. $\theta(a)\theta(s)/\theta(s) \in J_S\}$ are ideals of R and R' respectively. Therefore $\theta(I)$ is a principal ideal of R', $\theta(I) = (\theta(a))$ say. We prove $J_S = (\theta(a)/\theta(s))$ for some $s \in S$. For suppose $\theta(b)/\theta(t) \in J_S$. This implies $b/t \in I_S$ and $b/t\ ts/s = bts/ts \in I_S$. Therefore $b \in I$ and $\theta(b) = \theta(a)r'$ for some $r' \in R'$. Then

$\theta(b)/\theta(t) = \theta(a)/\theta(s)\ \theta(s)r'/\theta(t) \in (\theta(a)/\theta(s))$. Also we have $\theta(a)/\theta(s) = \theta(a)\theta(s)/\theta(s)\ 1/\theta(s) \in J_S$. Hence $J_S = (\theta(a)/\theta(s))$ and this shows that R_S is a PIR (wrt. $\theta' : R_S \rightarrow R'_{\theta(S)}$).

Here we give the results for direct sums and direct summands. First a finite direct sum of PIR's is a PIR.

Theorem 2.13: If R_i is a PIR (wrt. $\theta_i : R_i \rightarrow R'_i$) for i = 1, ..., n, then $R = R_1 \oplus \ldots \oplus R_n$ is a PIR (wrt. $\theta : R_1 \oplus \ldots \oplus R_n \rightarrow R'_1 \oplus \ldots \oplus R'_n$) where $\theta((r_1, \ldots, r_n)) = (\theta_1(r_1), \ldots, \theta_n(r_n))$.

Proof: Clearly θ is a G-monomorphism. Suppose $I' = \theta(I)$ is an ideal in $R'_1 \oplus \ldots \oplus R'_n$ for some ideal I in R. Therefore we can write $I' = I'_1 \oplus \ldots \oplus I'_n$ such that I'_i is an ideal of R'_i for i = 1, ..., n. Because, take

$I'_i = \{a'_i \in R'_i : \exists\ (a'_1, \ldots, a'_n) \in I'$, for $a'_j \in R'_j$, $i \neq j = 1, \ldots, n\}$.

If $(a'_1, \ldots, a'_n) \in I'$, then $a'_i \in I'_i$ for i = 1, ..., n. Therefore $(a'_1, \ldots, a'_n) \in I'_1 \oplus \ldots \oplus I'_n$.

Conversely if $(b'_1, \ldots, b'_n) \in I'_1 \oplus \ldots \oplus I'_n$, then $(a'_1, \ldots, b'_i, a'_{i+1}, \ldots, a'_n) \in I'$ for some $a'_j \in R'_j$ $(i \neq j = 1, \ldots, n)$. Since I' is an ideal of $R'_1 \oplus \ldots \oplus R'_n$, for $(0, \ldots, 0, 1_{R'_i}, 0, \ldots, 0) \in R'_1 \oplus \ldots \oplus R'_n$, $(a'_1, \ldots, a'_{i-1}, b'_i, a'_{i+1}, \ldots, a'_n)(0, \ldots, 0, 1_{R'_i}, 0, \ldots, 0) = (0, \ldots, 0, b'_i, 0, \ldots, 0) \in I'$, for i = 1, ..., n. Therefore $(b'_1, ..., b'_n) \in I'$ and hence $I' = I'_1 \oplus ... \oplus I'_n$. Now, consider $I' = \theta(I)$, where we can write $I = I_1 \oplus ... \oplus I_n$ such that I_i is an ideal of R_i for i = 1, ..., n. Because, take $I_i = \{a_i \in R_i : \exists (b_1,..., b_{i-1}, a_i, b_{i+1},..., b_n) \in I$, for some $b_j \in R_j$, $i\neq j = 1,..., n\}$. Hence if $(a_1, ..., a_n) \in I$ then $(a_1, ..., a_n) \in I_1 \oplus ... \oplus I_n$. Conversely suppose $(b_1, ..., b_n) \in I_1 \oplus ... \oplus I_n$, we have $(a_1, ..., b_i, a_{i+1},..., a_n) \in I$ for some $a_j \in R_j$ ($i\neq j = 1,..., n$). Since I is an ideal of R,

$(a_1, ..., b_i, a_{i+1},..., a_n)(0, ..., 0, 1_{R_i}, 0, ..., 0) = (0, ..., 0, b_i, 0, ..., 0) \in I$,

for i = 1, ..., n and $(b_1, ..., b_n) \in I$. Hence $I = I_1 \oplus ... \oplus I_n$. Therefore clearly

$\theta(I) = \theta(I_1 \oplus ... \oplus I_n) = \theta_1(I_1) \oplus ... \oplus \theta_n(I_n)$.

Now, we show that $I'_i = \theta_i(I_i)$ for i = 1, ..., n. For suppose $a'_i \in I'_i$, then there exist

$(b'_1, ..., b'_{i-1}, a'_i, b'_{i+1}, ..., b'_n) \in I' = \theta(I)$ for $i \neq j = 1, ..., n$. Therefore $a'_i \in \theta_i(I_i)$. Conversely if $\theta_i(b_i) \in \theta_i(I_i)$, then $b_i \in I_i$ and there exists $(a_1, ..., a_{i-1}, b_i, a_{i+1},..., a_n) \in I$ for some $a_j \in R_j$ ($i \neq j = 1, ..., n$). Hence

$\theta((a_1, ..., a_{i-1}, b_i, a_{i+1},..., a_n)) = (\theta_1(a_1), ..., \theta_i(b_i),..., \theta_n(a_n)) \in I' = I'_1 \oplus ... \oplus I'_n$.

This implies $\theta_i(b_i) \in I'_i$ and so $I'_i = \theta_i(I_i)$. Since R_i is a PIR wrt. $\theta_i : R_i \to R'_i$ for i =1,...,n, $I'_i = R'_i x_i$ for some $x_i \in R'_i$. Therefore $I' = I'_1 \oplus ... \oplus I'_n = R'_1 x_1 \oplus ... \oplus R'_n x_n = (R'_1 \oplus ... \oplus R'_n)(x_1, ..., x_n)$

and hence I' is a principal ideal of $R'_1 \oplus ... \oplus R'_n$. Hence $R_1 \oplus ... \oplus R_n$ is a PIR (wrt. $\theta : R_1 \oplus ... \oplus R_n \to R'_1 \oplus ... \oplus R'_n$).

Theorem 2.14: If R is a PIR (wrt. $\theta : R \to R'$) and $R \cong R_i \oplus ... \oplus R_n$, then for i = 1, ..., n each R_i is a PIR wrt. $\theta_i : R_i \to R'$ provided that $\theta_i(R_i)$ is an ideal of R', where $\theta_i(r_i) = \theta((0, ..., 0, r_i, 0, ..., 0))$.

Proof: Clearly θ_i is a G-monomorphism. Suppose I' is any ideal in R' such that $I' = \theta_i(I_i)$ for some ideal I_i in R_i. By the definition of θ_i we have

$\theta_i(I_i) = \theta((0) \oplus ... \oplus (0) \oplus I_i \oplus (0) \oplus ...\oplus (0)) = I'$.

Since R is a PIR wrt. $\theta : R \to R'$ then I' is principal in R'. Therefore R_i is a PIR with respect to $\theta_i : R_i \to R'$ for i = 1, ..., n.

We note that the restiriction of map $\theta : R \to R'$ to $\theta : R \to <\theta_i(R_i), 1>$ gives a G-monomorphism without the extra condition. Then the theorem falls. We give a counter-example. Consider $\theta : (O_2 \oplus O_2) \oplus O_2 \to Z_4 \oplus Z_4 \oplus Z_4$ given by

$\theta((x_1, x_2, x_3)) = \{(y_1, y_2, y_3) : y_i = 2$ if $x_i = a$ and $y_i = 0$ if $x_i = 0$ for i = 1, 2, 3}.

$O_2 \oplus O_2 \oplus O_2$ is a PIR (wrt. $\theta : O_2 \oplus O_2 \oplus O_2 \to Z_4 \oplus Z_4 \oplus Z_4$). Take $\theta_1 : O_2 \oplus O_2 \to Z_4 \oplus Z_4 \oplus Z_4$ given by $\theta_1((x_1, x_2)) = \theta((x_1, x_2, 0))$. Then $\theta_1 : O_2 \oplus O_2 \to <\theta_1(O_2 \oplus O_2), (1, 1, 1)>$ is a G-monomorphism, but $O_2 \oplus O_2$ is not a PIR (wrt. $\theta_1 : O_2 \oplus O_2 \to <\theta_1(O_2 \oplus O_2), (1, 1, 1)>$) since $\theta_1(O_2 \oplus O_2)$ is not a principal ideal of $<\theta_1(O_2 \oplus O_2), (1, 1, 1)>$.

Here we give some examples.

Example 2.15:

i. onsider the G-monomorphism $\theta : O_2 \oplus O_2 \to Z_4 \oplus Z_4$ given by $\theta((x_1, x_2)) = \{(y_1, y_2) : y_i = 2$ if $x_i = a$ and $y_i = 0$ if $x_i = 0$ for i = 1, 2}.
   Since O_2 is PIR (wrt. $\theta_1: O_2 \to Z_4$ given by $\theta_1(a) = 2$ and $\theta_1(0) = 0$), from Theorem 2.15 $O_2 \oplus O_2$ is a PIR (wrt. $\theta : O_2 \oplus O_2 \to Z_4 \oplus Z_4$)
ii. $O_2 \oplus O_2$ is not a PIR (wrt. $\theta : O_2 \oplus O_2 \to (O_2 \oplus O_2) \times Z$) since $\theta(O_2 \oplus O_2)$ is not a principal ideal of $(O_2 \oplus O_2) \times Z$.
iii. We easily see that O_2 is a PIR (wrt. $\theta_1 : O_2 \to O_2 \times Z$ given by $\theta_1(a) = (a, 0)$ and $\theta_1(0) = (0, 0)$). Therefore from Theorem 2.13, $O_2 \oplus O_2$ is a PIR (wrt. $\theta : O_2 \oplus O_2 \to (O_2 \times Z) \oplus (O_2 \times Z)$). We point out that $O_2 \times Z$ is not a PIR (wrt.1 : $O_2 \times Z \to O_2 \times Z$) since $\{(x, 2k) : x \in O_2$ and $k \in Z\}$ is an ideal of $O_2 \times Z$ but not principal.

We study the quotient rings of PIR's. Here we have the following result.

Theorem 2.16: If R is a PIR (wrt. $\theta : R \to R'$) and I ($\neq$ R) is an ideal of R such that $\theta(I)$ is an ideal of R' then R / I is a PIR (wrt. $\theta': R / I \to R'/ \theta(I)$ given by $\theta'(r + I) = \theta(r) + \theta(I)$).

Proof: Clearly θ' is a G-monomorphism. Let J' be an ideal of $R'/ \theta(I)$ such that $J' = \theta'(J)$ where J is an ideal of R / I. Then we can assume that $J' = J'_0 / \theta(I)$ where J'_0 is an ideal of R' which contains $\theta(I)$ and similarly $J = J_0 / I$ where J_0 is an ideal of R which contains I (Hungerford, 1974). By the definition of θ', it is clear that

$\theta'(J_0 / I) = \theta(J_0)/\theta(I)$.

We prove that $\theta(J_0)$ is an ideal of R'. For suppose $\theta(j_1)$, $\theta(j_2) \in \theta(J_0)$, then

$\theta(j_1) - \theta(j_2) = \theta(j_1 - j_2)$, since J_0 is an ideal of R, $j_1 - j_2 \in J_0$ and hence $\theta(j_1) - \theta(j_2) \in \theta(J_0)$. If $\theta(j_0) \in \theta(J_0)$, $r' \in R'$ then $r' + \theta(I) \in R'/\theta(I)$ and $\theta(j_0) + \theta(I) \in \theta(J_0)/\theta(I) = \theta'(J) = J'$ and this implies $(\theta(j_0) + \theta(I))(r' + \theta(I)) \in J'$. Therefore $r'\theta(j_0) + \theta(I) \in \theta(J_0)/\theta(I)$ and hence $r'\theta(j_0) \in \theta(J_0)$ i.e $\theta(J_0)$ is an ideal of R'. Since R is a PIR (wrt. $\theta : R \to R'$), then $\theta(J_0)$ is a

principal ideal in R'. Therefore for $\theta(a) \in R'$, $\theta(J_0) = (\theta(a))$ say. Now, we show that $J' = (\theta(a) + \theta(I))$. For take any element $\theta(j_0) + \theta(I) \in J'$ where $j_0 \in J_0$, then we have in R', $\theta(j_0) = r'\theta(a)$ for some $r' \in R'$. Therefore $\theta(j_0) + \theta(I) = r'\theta(a) + \theta(I) = (r' + \theta(I))(\theta(a) + \theta(I)) \in (\theta(a) + \theta(I))$.
Suppose $(r' + \theta(I))(\theta(a) + \theta(I)) \in (\theta(a) + \theta(I))$. Then $(r' + \theta(I))(\theta(a) + \theta(I)) = r'\theta(a) + \theta(I)$. Since $\theta(J_0)$ is an ideal of R', $r'\theta(a) \in \theta(J_0)$. Therefore $r'\theta(a) + \theta(I) \in \theta(J_0)/\theta(I) = J'$. Hence J' is a principal ideal of $R'/\theta(I)$ and $J' = (\theta(a) + \theta(I))$. Therefore R / I is a PIR (wrt. $\theta' : R / I \rightarrow R'/ \theta(I)$).
We extend the result "a Euclidean ring is a PIR (Fletcher,1971) " to the case where R does not necessarily have an identity.

Theorem 2.17: Every Euclidean ring (in the sense of Group A) is a PIR.

Proof: Suppose R is a Euclidean ring (according to Definition 1.6 of Group A) wrt. $\theta : R \rightarrow R'$. Take any ideal J in R' such that $J = \theta(I)$ for some ideal I in R. Let us show that J is principal. If $I = (0)$ then $J = (0)$ and hence J is principal. Suppose now I is a non-zero ideal in R, then it contains non-zero elements, choose $0 \neq b \in I$ such that
$\varphi(b) = \min.\{\varphi(x) : x \neq 0, x \in I\}$ where $\varphi : R \rightarrow W$ is an algorithm for R.
If $a \in I$, then there exist $q' \in R'$ and $r \in R$ such that
$\theta(a) = \theta(b)q' + \theta(r)$ with $r = 0$ or $\varphi(r) < \varphi(b)$.
Since $\theta(a) \in \theta(I)$ and $q'\theta(b) \in (\theta(b)) \subset \theta(I)$, $\theta(r)$ is necessarily in $\theta(I) = J$ i.e. $r \in I$. Since $\varphi(r) < \varphi(b)$ is a contradiction, we must have $r = 0$. Consequently
$\theta(I) \subseteq (\theta(b)) \subseteq \theta(I)$. Therefore $J = (\theta(b))$ in R' and R is a PIR (wrt. $\theta : R \rightarrow R'$).
Note that here it is straightforward to check that if the well-ordered set is taken to be $Z^{+} \cup \{0\}$ then we have immediately Theorem 2.17 in the sense of Group B and Group C.

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