



Journal of Applied Sciences

ISSN 1812-5654

science
alert

ANSI*net*
an open access publisher
<http://ansinet.com>

A Technique to Avoid Divergence for the Planar and Spatial Newton-homotopy Continuation Method

Tzong-Mou Wu

Department of Mechanical Engineering, Far East College, Tainan 744, Taiwan

Abstract: The traditional Newton method is a well-known and popular method for solving non-linear equation. It has high efficient in the convergence speed. However, we always need to guess the initial value in the iteration process. Good initial guess value can solve the equation quickly. Bad initial guess value usually will yield divergence. Homotopy continuation method is a kind of perturbation method. It can guarantee the answer by a certain path if we choose the auxiliary homotopy functions, or call the start system, well. This study presents some useful techniques for the choice of the auxiliary homotopy functions to avoid divergence and time-consuming computation when solving the planar and spatial simultaneous non-linear equations.

Key words: Numerical method, Newton method, homotopy continuation method

INTRODUCTION

Homotopy continuation method was known as early as in the 1930's. This method was used by kinematician in the 1960's at U.S. for solving mechanism synthesis problems. The latest development was done by Morgan^[1,2] at GM. We also have two important literatures: Garcia^[3] and Allgower^[4]. Continuation method gives a set of certain answers and some simple iteration process to obtain our solutions more exactly. Wu^[6] provided some rules for the choice of the auxiliary homotopy functions to avoid divergence of traditional Newton method for one-dimensional non-linear algebraic equation. Wu^[5] also applied the homotopy continuation method to search all the roots of inverse kinematics problem of robot and obtained more but not all convergence answers than the traditional numerical methods. Therefore, present research attempts to study some robust rules and provide the choice method of the auxiliary homotopy functions for homotopy continuation method to avoid the problem of divergence of traditional Newton method when solving the non-linear equations.

NEWTON-HOMOTOPY CONTINUATION METHOD

Before the study of the auxiliary homotopy functions, we consider the following simultaneous non-linear equations

$$F(X)=0 \quad (1)$$

To solve the equations system, we have many different methods such as Newton-Raphson method, etc.

As we known, the numerical iteration formula of Newton's method for solving the equations system was given as:

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} -F_1(x_{1n}, x_{2n}, \dots, x_{nn}) \\ -F_2(x_{1n}, x_{2n}, \dots, x_{nn}) \\ \vdots \\ -F_n(x_{1n}, x_{2n}, \dots, x_{nn}) \end{bmatrix} \quad (2)$$

Observing Eq. 2, we find that when the determinant

$$\text{Det}(F) \equiv \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{vmatrix} = 0 \text{ or } \rightarrow 0 \quad (3)$$

The $[\Delta x_1, \Delta x_2, \dots, \Delta x_n]^T$ values will become infinite. Now, we will apply the homotopy continuation method and develop some useful techniques for the choice of the auxiliary homotopy functions to avoid divergence of iteration results in Eq. 2.

Given a set of equations in n variables $x_1, x_2, \dots, x_n$. We modify the equations by omitting some of the terms and adding new ones until we have a new system of equations, the solutions to which may be easily

guessed/given/known. We then deform the coefficients of the new system into the coefficients of the original system by a series of small increments and we follow the solution through the deformation, using methods such as Newton-Raphson. This is called Newton-homotopy continuation original system.

If we wish to find the solution vector in Eq. 1, we choose a new simple start system or call the auxiliary homotopy functions:

$$F'(X)=0 \quad (4)$$

The auxiliary homotopy functions $F'(X)$ must be known or controllable and easy to solve. Then, we define the homotopy continuation as:

$$H(X,t)=tF(X)+(1-t)F'(X)=0 \quad (5)$$

Where t is an arbitrary parameter and varies from 0 to 1, i.e. $t \in [0,1]$. Therefore, we have the following two boundary conditions

$$\begin{aligned} H(X,0) &= F'(X) \\ H(X,1) &= F(X) \end{aligned} \quad (6)$$

This is the famous homotopy continuation method. It is also called Bootstrap method or Parameter-Perturbation method, but these names did not become popular.

Our goal in this study is to solve $H(X,t) = 0$ instead of $F(X) = 0$ by varying parameter t from 0 to 1 and avoid the situation of divergence. So we rewrite Eq. 2 as:

$$\begin{bmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \dots & \frac{\partial H_1}{\partial x_n} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \dots & \frac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial H_n}{\partial x_1} & \frac{\partial H_n}{\partial x_2} & \dots & \frac{\partial H_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} -H_1(x_{1n}, x_{2n}, \dots, x_{nn}) \\ -H_2(x_{1n}, x_{2n}, \dots, x_{nn}) \\ \vdots \\ -H_n(x_{1n}, x_{2n}, \dots, x_{nn}) \end{bmatrix} \quad (7)$$

Where the situation of divergence still may occurs at

$$\text{Det}(H) = \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \dots & \frac{\partial H_1}{\partial x_n} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \dots & \frac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial H_n}{\partial x_1} & \frac{\partial H_n}{\partial x_2} & \dots & \frac{\partial H_n}{\partial x_n} \end{vmatrix} = 0 \text{ or } \rightarrow 0 \quad (8)$$

To avoid divergence, we have to let $\text{Det}(H) \neq 0$ or $\rightarrow 0$. In this study, we develop a new technique for this purpose.

Planar problem: Firstly, consider the planar problem

$$\begin{cases} f(x_1, x_2) = 0 \\ g(x_1, x_2) = 0 \end{cases} \Rightarrow \begin{cases} H_1 = tf + (1-t)f' = 0 \\ H_2 = tg + (1-t)g' = 0 \end{cases} \quad (9)$$

The determinant $\text{Det}(H)$ therefore is

$$\begin{aligned} \text{Det}(H) &= \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{vmatrix} = \frac{\partial H_1}{\partial x_1} \times \frac{\partial H_2}{\partial x_2} - \frac{\partial H_1}{\partial x_2} \times \frac{\partial H_2}{\partial x_1} \\ &= (tf_1 + (1-t)f'_1)(tg_2 + (1-t)g'_2) - \\ &\quad (tf_2 + (1-t)f'_2)(tg_1 + (1-t)g'_1) \\ &= t^2(A - B + C) + t(B - 2C) + C \end{aligned} \quad (10)$$

Where,

$$\begin{aligned} A &= f_1g_2 - f_2g_1 \\ B &= f_1g'_2 + f'_1g_2 - f_2g'_1 - f'_2g_1 \\ C &= f'_1g'_2 - f'_2g'_1 \end{aligned} \quad (11)$$

If we let $A-B+C = 0$ and $B-2C = 0$, we can avoid many situations of divergence and economize the computation time. This technique yields

$$\begin{aligned} \begin{cases} B = 2C \\ A = C \end{cases} \\ \Rightarrow B = A + C \\ \Rightarrow f_1g'_2 + f'_1g_2 - f_2g'_1 - f'_2g_1 = f_1g_2 - f_2g_1 + f'_1g'_2 - f'_2g'_1 \\ \Rightarrow (f_1 - f'_1)(g'_2 - g_2) = (f_2 - f'_2)(g'_1 - g_1) \end{aligned} \quad (12)$$

To satisfy this equation, we have four cases. They are

$$\begin{aligned} \text{Case 1: } & f_1 - f'_1 = 0 \wedge f_2 - f'_2 = 0 \\ \text{Case 2: } & g'_2 - g_2 = 0 \wedge g'_1 - g_1 = 0 \\ \text{Case 3: } & f_1 - f'_1 = 0 \wedge g'_1 - g_1 = 0 \\ \text{Case 4: } & g'_2 - g_2 = 0 \wedge f_2 - f'_2 = 0 \end{aligned} \quad (13)$$

But case 1 or case 2 will yield useless result, i.e. $f' = f, g' = g$. Since the auxiliary homotopy functions f' and g' must be different from the original functions f and g . Hence we just can apply case 3 or case 4. If we choose case 3, then we have

$$\begin{cases} f'_1 = f_1 \\ g'_1 = g_1 \end{cases} \Rightarrow \begin{cases} f' = f + p \\ g' = g + q \end{cases} \quad (14)$$

Substituting this equation into $A = C$, we obtain the following formula for the choice of the auxiliary homotopy functions f' and g'

$$\begin{aligned}
 A &= C \\
 \Rightarrow f_1 g_2 - f_2 g_1 &= f'_1 g'_2 - f'_2 g'_1 \\
 \Rightarrow f_1 g_2 - f_2 g_1 &= f_1 g'_2 - f'_2 g_1 \\
 \Rightarrow f_1 (g_2 - g'_2) &= g_1 (f_2 - f'_2) \\
 \Rightarrow f_1 (-q_2) &= g_1 (-p_2) \\
 \Rightarrow \begin{cases} p_2 = f_1 \\ q_2 = g_1 \end{cases} &\Rightarrow \begin{cases} p = \int f_1 dx_2 \\ q = \int g_1 dx_2 \end{cases} \\
 \Rightarrow \begin{cases} f' = f + \int f_1 dx_2 \\ g' = g + \int g_1 dx_2 \end{cases}
 \end{aligned} \quad (15)$$

So we conclude that

$$\begin{cases} H_1 = tf + (1-t)f' = tf + (1-t) \\ \quad (f + \int f_1 dx_2) = f + (1-t) \int f_1 dx_2 \\ H_2 = tg + (1-t)g' = tg + (1-t) \\ \quad (g + \int g_1 dx_2) = g + (1-t) \int g_1 dx_2 \end{cases} \quad (16)$$

By this way, the determinant $\text{Det}(H)$ becomes only $f'_1 g'_2 - f'_2 g'_1$. Therefore, we only need to judge and avoid this form not equal to zero or not approach to infinite to guarantee the convergence.

Spatial problem: Secondly, consider the spatial problem

$$\begin{cases} f(x_1, x_2, x_3) = 0 \\ g(x_1, x_2, x_3) = 0 \\ h(x_1, x_2, x_3) = 0 \end{cases} \Rightarrow \begin{cases} H_1 = tf + (1-t)f' = 0 \\ H_2 = tg + (1-t)g' = 0 \\ H_3 = th + (1-t)h' = 0 \end{cases} \quad (17)$$

The determinant $\text{Det}(H)$ is

$$\text{Det}(H) = \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \\ \frac{\partial H_3}{\partial x_1} & \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} \quad (18)$$

We would like to apply the character developed in planar problem, so we must expand the $\text{Det}(H)$ as the combination of three two-order forms.

$$\begin{aligned}
 \text{Det}(H) &= \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \\ \frac{\partial H_3}{\partial x_1} & \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} = \frac{\partial H_1}{\partial x_1} \begin{vmatrix} \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \\ \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} \\
 &\quad - \frac{\partial H_2}{\partial x_1} \begin{vmatrix} \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} + \frac{\partial H_3}{\partial x_1} \begin{vmatrix} \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \end{vmatrix}
 \end{aligned} \quad (19)$$

By the same manner, the three two-order forms can be expressed as:

$$\begin{aligned}
 \begin{vmatrix} \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \\ \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} &= t^2 (A_1 - B_1 + C_1) + t(B_1 - 2C_1) + C_1 \\
 \begin{vmatrix} \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} &= t^2 (A_2 - B_2 + C_2) + t(B_2 - 2C_2) + C_2 \\
 \begin{vmatrix} \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \end{vmatrix} &= t^2 (A_3 - B_3 + C_3) + t(B_3 - 2C_3) + C_3
 \end{aligned} \quad (20)$$

Where:

$$\begin{aligned}
 A_1 &= g_2 h_3 - g_3 h_2 \\
 B_1 &= g_2 h'_3 + g'_2 h_3 - g_3 h'_2 - g'_3 h_2 \\
 C_1 &= g'_2 h'_3 - g'_3 h'_2 \\
 A_2 &= f_2 h_3 - f_3 h_2 \\
 B_2 &= f_2 h'_3 + f'_2 h_3 - f_3 h'_2 - f'_3 h_2 \\
 C_2 &= f'_2 h'_3 - f'_3 h'_2 \\
 A_3 &= f_2 g_3 - f_3 g_2 \\
 B_3 &= f_2 g'_3 + f'_2 g_3 - f_3 g'_2 - f'_3 g_2 \\
 C_3 &= f'_2 g'_3 - f'_3 g'_2
 \end{aligned} \quad (21)$$

Similarly, we let $A_i - B_i + C_i = 0$ and $B_i - 2C_i = 0$, $i = 1, 2, 3$. We can also obtain the following formula for the choice of the auxiliary homotopy functions f' , g' and h'

$$\begin{cases} f' = f + \int f_2 dx_3 \\ g' = g + \int g_2 dx_3 \\ h' = h + \int h_2 dx_3 \end{cases} \quad (22)$$

And

$$\begin{cases} H_1 = tf + (1-t)f' = tf + (1-t) \\ (f + \int f_2 dx_3) = f + (1-t) \int f_2 dx_3 \\ H_2 = tg + (1-t)g' = tg + (1-t) \\ (g + \int g_2 dx_3) = g + (1-t) \int g_2 dx_3 \\ H_3 = th + (1-t)h' = th + (1-t) \\ (h + \int h_2 dx_3) = h + (1-t) \int h_2 dx_3 \end{cases} \quad (23)$$

Also the determinant $\text{Det}(H)$ will become

$$\text{Det}(H) = \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \frac{\partial H_1}{\partial x_3} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \frac{\partial H_2}{\partial x_3} \\ \frac{\partial H_3}{\partial x_1} & \frac{\partial H_3}{\partial x_2} & \frac{\partial H_3}{\partial x_3} \end{vmatrix} = \frac{\partial H_1}{\partial x_1} (g'_2 h'_3 - g'_3 h'_2) - \frac{\partial H_2}{\partial x_1} (f'_2 h'_3 - f'_3 h'_2) + \frac{\partial H_3}{\partial x_1} (f'_2 g'_3 - f'_3 g'_2) \quad (24)$$

Above are the techniques for choosing the appropriate auxiliary homotopy functions $F'(X)$. In the following section, we will run two examples to compare the improved Newton-homotopy method and traditional Newton method.

Example 1: Consider a planar robot defined by Wu^[5]:

$$\begin{cases} f(\theta_1, \theta_2) = L_1 C\theta_1 + L_2 C\theta_2 - Px = 0 \\ g(\theta_1, \theta_2) = L_1 S\theta_1 + L_2 S\theta_2 - Py = 0 \end{cases} \quad (25)$$

Where, $C = \cos$, $S = \sin$. L_1, L_2, Px and Py are givens and assuming that $L_1 = 2, L_2 = 1, Px = -1, Py = 1.5$. θ_1 and θ_2 are unknowns. By the formula, we have

$$\begin{cases} H_1 = f + (1-t) \int f_1 d\theta_2 = L_1 C\theta_1 + L_2 C\theta_2 - \\ Px + (1-t)(-L_1 S\theta_1)\theta_2 = 0 \\ H_2 = g + (1-t) \int g_1 d\theta_2 = L_1 S\theta_1 + L_2 S\theta_2 - \\ Py + (1-t)(L_2 C\theta_1)\theta_2 = 0 \end{cases} \quad (26)$$

The divergence condition $f'_1 g'_2 - f'_2 g'_1$ in this example is:

$$f'_1 g'_2 - f'_2 g'_1 = -L_1 L_2 (\sin(\theta_1 - \theta_2) - \cos(\theta_1 - \theta_2)\theta_2) - L_1^2 \theta_2 \quad (27)$$

Since the domain $\theta_1, \theta_2 \in [0^\circ, 360^\circ]$, we have the singular points $(\theta_1, \theta_2) = (0^\circ, 0^\circ), (180^\circ, 0^\circ)$ or $(360^\circ, 0^\circ)$ in the above equation.

Table 1: The number of divergence of the traditional Newton and the improved Newton-homotopy method

	Planar problem		Spatial problem	
	Newton	Homotopy	Newton	Homotopy
$d\theta = 10^\circ$	77	0	2035	0
$d\theta = 1^\circ$	725	0	136819	0

Solving Eq. 26 by the Newton-homotopy iteration formula, Eq. 7 and avoiding the three singular points, we obtain the number of divergence of the traditional Newton and the improved Newton-homotopy method, respectively in Table 1.

Observing Table 1, we find that the number of divergence will increase with the initial guesses increasing in the traditional Newton method. Whereas, we can guarantee the convergence whatever how many initial guesses we have in the improved Newton-homotopy method just only the singular points being eliminated.

Example 2: Consider a spatial robot defined by Wu^[5]:

$$\begin{cases} f(\theta_1, \theta_2, S) = C\theta_1 S\theta_2 S - S\theta_1 L - Px = 0 \\ g(\theta_1, \theta_2, S) = S\theta_1 S\theta_2 S + C\theta_1 L - Py = 0 \\ h(\theta_1, \theta_2, S) = C\theta_2 S - Pz = 0 \end{cases} \quad (28)$$

Where, L, Px, Py and Pz are givens and their values are assumed that $L = 2, Px = -2, Py = 1.5, Pz = 1$. θ_1, θ_2 and S are unknowns, their domains in this example are $\theta_1, \theta_2 \in [0^\circ, 360^\circ]$ and $S \in [-3, 3]$, respectively. Using formula (23), we have

$$\begin{cases} H_1 = f + (1-t) \int f_2 dS = C\theta_1 S\theta_2 S - S\theta_1 L - \\ Px + (1-t)(\frac{1}{2}C\theta_1 C\theta_2 S^2) = 0 \\ H_2 = g + (1-t) \int g_2 dS = S\theta_1 S\theta_2 S + C\theta_1 L - \\ Py + (1-t)(\frac{1}{2}S\theta_1 C\theta_2 S^2) = 0 \\ H_3 = h + (1-t) \int h_2 dS = C\theta_2 S - \\ Pz + (1-t)(-\frac{1}{2}S\theta_2 S^2) = 0 \end{cases} \quad (29)$$

According to Eq. 24, the determinant $\text{Det}(H)$ is

$$\text{Det}(H) = S(1 + \frac{1}{2}S^2)(\frac{1}{2}(t-1)(C\theta_2 S^2) - S\theta_2 S) \quad (30)$$

By means of the domains of θ_1, θ_2 and S , we obtain the singular points occur at initial condition $t = 0$, i.e. initial singular point guesses, are $S = 0$ or $S = -2\tan\theta_2$.

Similarly, solving Eq. 29 by the Newton-homotopy iteration formula, Eq. 7, and avoiding these initial singular point guesses, we obtain the number of divergence of the traditional Newton and the improved Newton-homotopy method respectively in Table 1 ($dS = 1$).

Clearly, we also find the use of traditional Newton method will easily become divergence when the bad initial guesses been chosen. However, this situation can be avoided in the improved Newton-homotopy continuation method.

CONCLUSIONS

Typically, it is usually a big trouble and disadvantage for us to do the algebraic operation, for example, solving the non-linear equations. Fortunately, by the aid of computer science, the non-linear equations will be solved no more difficulty. With the computer improved and the numerical technique developed, the solutions of non-linear equations become easier. We can use current high-speed processor to determine the solutions quickly.

Although, we already have many different numerical methods can help us to treat these non-linear equations. The traditional Newton-Raphson method is a popular numerical method. However, it still has some well-known critical defects. By means of the development of numerical continuation technique, the defects in traditional numerical methods such as the acquirement of a good initial guess, the problems of convergence and the computing time will be avoided. Homotopy continuation method is one of the famous improved numerical continuation methods. Its convergence speed is fast, also the algorithm is clear and easy.

This study combines Newton's and homotopy continuation method and proposes some robust rules for the choice of the auxiliary homotopy function for the Newton-homotopy continuation method to avoid the problem of divergence of many traditional numerical methods when solving the non-linear equations. We can get well outcome by this new method. The new interesting method will provide another numerical approach. It is hoped that the study presented here will contribute towards progress in the numerical techniques and other relative fields for scientists or engineers.

REFERENCES

1. Morgan, A.P., 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. GM Research Publication, GMR 3651.
2. Morgan, A.P., 1987. Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems. Prentice-Hall Book Company, Inc., Englewood Cliffs, New Jersey.
3. Garcia, C.B. and W.I. Zangwill, 1981. Pathways to Solutions, Fixed Points and Equilibria. Prentice-Hall Book Company, Inc., Englewood Cliffs, New Jersey.
4. Allgower, E.L. and K. Georg, 1990. Numerical Continuation Methods, An Introduction. Springer-Verlag, New York.
5. Wu, T.M., 2005. Searching all the roots of inverse kinematics problem of robot by homotopy continuation method. *J. Applied Sci.*, 5: 666-673.
6. Wu, T.M., 2005. A study of convergence on the Newton-homotopy continuation method. *Applied Mathematics and Computation* (In Press).