Journal of Applied Sciences 5 (1): 18-23, 2005
ISSN 1607-8926

On Multi-valued Semantics for Logic Programs

Victor Felea
Faculty of Computer Science, University "AL.I.Cuza" Iasi, General Berthelot 16, Iasi, 6600, Romania

Abstract: For a general program P, multi-valued interpretations and models are defined, considering a set of truth logic values and an undefined value. The program P may contain constant propositions, which are defined for each truth logic value. Two orderings between the set of all multi-valued interpretations are considered: one is Fitting ordering and the other is standard ordering. The semantics of type well-founded and of type stable for a program P are introduced. This study showed that the well-founded model is the least stable model with respect to Fitting ordering.

Key words: Fixed points, stable models, well-founded models

INTRODUCTION

The well-founded semantics has been introduced by Van Gelder *et al.*[1]. It is a 3-valued semantics. They use as truth values "true", "false" and "⊥" (an unknown truth value). They have shown that if a logic program P has a 2-valued well-founded model, then this model is the unique stable model of P.

The stable model semantics has been introduced by Gelfond and Lifschitz[2] and by Bidoit and Froidevaux[3].

Przymusinski[4] has introduced 3-valued stable models as a generalization of 2-valued stable models. He also found that the well-founded model of any program P coincides with the smallest 3-valued stable model of P.

Luong[5] has defined a new semantics for Datalog programs, which includes the well-founded models and all stable models.

Fitting[6] has studied the structure of the family of all stable models for a logic program using two orderings; one is called the knowledge ordering based on degree of definedness, the other is called truth ordering based on degree of truth. In the first ordering every logic program has a smallest stable model, which coincides with the well-founded model.

Przymusinski[7] has introduced the stable model semantics for disjunctive logic programs and deductive databases. For normal programs, the partial disjunctive stable semantics coincides with the well-founded semantics.

Loyer and Umberto[8] proposed a well-founded semantics for deductive databases with uncertainty frameworks.

Malfon[9] gives a new characterization of Fitting model and of the well-founded model.

Lallouet[10] has defined a semantics for normal logic programs based on the property of composition. This semantics extends well-founded semantics and Fitting semantics.

This study defines a semantics of type well-founded and a stable semantics for the case multi-valued interpretations and points out a relationship between them.

Interpretations and models: Let P be a general logic program in sense Gelder[1]. Let H be the Herbrand base associated to P. We consider a total ordered set of truth values $L_n = (0, v_1,..,v_{n-1},1)$, where, value 0 corresponds to false, value 1 is for true and the values v_k, $1 \leq k \leq n-1$ are intermediate between false and true. For every truth value v from L_n, we used a constant proposition denoted by c_v and defined by $c_v(A) = v$ for every ground atom A from H. The undefined value will be denoted by u and corresponding constant by c_u, where, $c_u(A) = u$ for every $A \in H$. Let us denote 0 by v_0 and 1 by v_1. The constant propositions c_v, as well c_u may appear in the bodies of rules from P.

Definition 1: By a multi-valued Herbrand partial interpretation I we mean a partial function from H into L_n. For an interpretation I, let us denote by V_I the vector of sets from H:

$$V_I = (S_0, S_1,...,S_n)$$

where, $S_j = \{A/A \in H \text{ and } I(A) = v_j\}$, $0 \leq j \leq n$.

We denote by S_u the set of all remaining atoms from $H - \bigcup_{j=0}^{n} S_j$. If $V = (S_0, S_1,...,S_n)$ where, S_j are disjoint sets of H, $0 \leq j \leq n$, then there is an interpretation I, such that $V_I = V$. In the case S_u is empty, then I is called total interpretation.

Corresponding Author: Victor Felea, Faculty of Computer Science, University "AL.I.Cuza" Iasi, General Berthelot 16, Iasi, 6600, Romania E-mail: felea@thor.infoiasi.ro

Assume that L_n admits a negation, denoted $\rceil$, which satisfies the following properties: $\rceil 0 = 1$, $\rceil 1 = 0$ and $v_i < v_j$ implies $\rceil v_j < \rceil v_i$, for every i,j, $0 \le i,j \le n$. Moreover, we consider $\rceil u = u$.

For an atom A, such that $A \in S_u$, we write I(A) = u and I(A) ≠ u, otherwise.

For a ground instantiated rule r of P, having the form: $r \equiv A \leftarrow L_1,..., L_m$ let us denote by $\hat{I}$ (body (r)) = min $\{\hat{I}(L_i), \hat{I}(L_i) \neq u, 1 \le i \le m\}$. We consider min ∅=1, where ∅ is empty set.

Let M_A be the set of all ground instantiated rule of P, having A as its head. Let $v^I_{j,A}$ be the truth value from L_n defined by: $v^I_{j,A} = \max\{\hat{I}(body(r)) / r \in M_A\}$.

The interpretation I is extended to ground literals denoted $\hat{I}$ by : $\hat{I}(A) = I(A)$ and $\hat{I}(\sim A) = \rceil I(A)$ for every ground atom $A \in H$.

In the following we define the notion of model for P.

Definition 2: An interpretation I satisfies the ground instantiated rule of P having the form $A \leftarrow L_1,...,L_m$ if one of the following relations holds:

a. there is j, $1 \le j \le m$, such that $\hat{I}(L_j) = 0$ or
b. $\hat{I}(A) \neq u$ and $\min\{\hat{I}(L_i), \hat{I}(L_i) \neq u, 1 \le i \le m\} \le \hat{I}(A)$ or
c. $\hat{I}(A) = u \Rightarrow [(\hat{I}(body(r)) = v^I_{j,A}) \Rightarrow (\exists i, 1 \le i \le m$, such that $\hat{I}(L_i) = u)]$.

An interpretation I is a model for P if I satisfies every ground instantiated rule of P.

In the following we need to specify two ordering between interpretations. The first one denoted $\le_F$ is of type Fitting and the second one denoted $\le_s$ is of type standard[4].

Definition 3: Let I and J be two interpretations, such that $V_I = (S_0,..., S_n)$ and $V_J = (T_0,..., T_n)$. We say that $I \le_F J$ if $S_j \subseteq T_j$, for every j, $0 \le j \le n$.

We say that $I \le_s J$ if $T_0 \subseteq S_0$, $S_n \subseteq T_n$ and $S_j \subseteq T_j \cup ... \cup T_{n-1}$ for every j, $1 \le j \le n-1$.

Remark 1: In the case n = 1, the ordering $\le_F$ is the Fitting ordering and $\le_S$ is the standard ordering. These orderings were used by Przymusinski[4] to study the well-founded semantics and three-valued stable semantics.

Stable semantics: Study defines here multi-valued stable models. Firstly, define an operator between the set of all interpretations of the program P. This operator will be denoted by S_P.

Definition 4: Let P be a logic program and I be an interpretation of P. We define the interpretation $S_P(I)$ in the following manner: if $V_{S_{P(I)}} = (T_0, T_1, ..., T_n)$ then:

i. For a ground atom A, $A \in T_0$ if for every ground instantiated rule of P, having the form $A \leftarrow L_1,..., L_m$, there exists i, $1 \le i \le m$, such that $\hat{I}(L_i) = 0$.
ii. For each h, $1 \le h < n$, a ground atom A is considered in T_h if a) and b) hold:
a. for every ground instantiated rule of P having the form: $A \leftarrow L_1,..., L_m$ we have: min $\{\hat{I}(L_j), \hat{I}(L_j) \neq u, 1 \le j \le m\} \le v_h$.
b. there is a ground instantiated rule of P of the form: $A \leftarrow V_1,..., V_m$, such that: min $\{\hat{I}(V_j), \hat{I}(V_j) \neq u, 1 \le j \le m\} = v_h$.
iii. For a ground atom A, A is considered in T_n, if there is a ground instantiated rule of P having the form $A \leftarrow C_1,..., C_q$, such that $\hat{I}(C_j) = 1$ for every j, $1 \le j \le q$.

Proposition 2: Let P be a positive program. The operator S_P as it is defined in the definition 4 is monotonic with respect to the standard ordering $\le_s$.

The proof results from the definition of the operator S_P and the standard ordering $\le_s$.

The existence of the least model with respect to $\le_s$ for a positive program P is emphasized by the following theorem:

Theorem 1: For a positive program P, there is the least fixed point of the operator S_P with respect to the ordering $\le_s$, denoted L_P. Moreover, L_P is the least model of P with respect to the ordering $\le_s$.

Proof: Consider $\perp = (H, \emptyset,..., \emptyset)$ the least interpretation with respect the ordering $\le_s$. The model L_P is obtained applying the operator S_P ω times: $\perp, S_P(\perp),...., S^\omega_P, (\perp)$ where, ω is the first ordinal.

The rest of the proof is classical, therefore it is skipped.

Now, we need to introduce an operator Γ* defined on the set of all interpretations, which extends the operator Γ defined by Przymusinski[4].

Definition 5: Let P be a general logic program and I an interpretation. We denote by P|I the positive program, which is obtained from P by replacing in every ground instantiated clause of P, all negative literals of the form ~A by c_v if I(A) ≠ u and by u otherwise, where $v = \rceil I(A)$. The program P|I is positive, hence applying the Theorem 1, it results that P|I admits a unique least model J with respect ordering $\le_s$. The operator Γ* is defined by: Γ*(I) = J.

Proposition 3: Let M be a fixed point of the operator Γ* from the definition 5. Then M is a minimal model of P with respect to ordering $\le_s$.

Proof: Let M be a fixed point of Γ^*, hence M is the least model of P|M with respect to $\leq_s$-ordering. Firstly, we show that M is a model for P. Let r be an arbitrary ground instantiated clause from P of the form:

$$r \equiv A \leftarrow B_1,..., B_p \sim D_1,..., \sim D_q \quad (1)$$

The corresponding clause r′ from P|M has the form:

$$r' \equiv A \leftarrow B_1,\dots,B_p, c_{v_1},\dots,c_{v_q} \quad (2)$$

where, $v_j = \rceil M(D_j)$ if $M(D_j) \neq u$ and u otherwise.

It results that $M(\sim D_j) = \rceil M(D_j) = c_{v_j}$, therefore M satisfies r′ iff M satisfies r.

Secondly, it must show that M is a minimal model for P with respect to $\leq_s$-ordering. Let M_1 be a model for P, such that $M_1 \leq_s M$. It is sufficient to show that M_1 is also a model for P|M. Since M is the least model of P|M with respect to $\leq_s$-ordering, we obtain that $M_1 \leq_s M$, hence $M_1 = M$.

For the ground instantiated clause r having the form (1), let us denote by r″ the corresponding clause to r from $P|M_1$:

$$r'' \equiv A \leftarrow B_1,\dots,B_p, c_{w_1},\dots,c_{w_q} \quad (3)$$

where, $w_j = \rceil M_1(D_j)$, for every j, $1 \leq j \leq q$.

As before, since M_1 is a model for P, it obtains that M_1 is a model for $P|M_1$, hence M_1 satisfies r″.

Since $M_1 \leq_s M$ and using the definition of v_j and w_j, the following statements are satisfied:

i. if $w_j = 0$ then $v_j = 0$, for every j, $1 \leq j \leq q$.
ii. if $v_j = 1$ then $w_j = 1$, $1 \leq j \leq q$.
iii. if $w_j = 1$ then we have: $v_j \leq w_j$ whenever $v_j \neq u$, $1 \leq j \leq q$.
iv. If $0 < w_j < 1$, then we have ($v_j \neq u$ and $v_j \leq w_j$, $1 \leq j \leq q$).

These statements imply the inequality:

$$\min\{M_1(B_i), M_1(B_i) \neq u, 1 \leq i \leq p, c_{v_j}, v_j \neq u, 1 \leq j \leq q\} \leq$$

$$\leq \min\{M_1(B_i), M_1(B_i) \neq u, 1 \leq i \leq p, c_{w_j}, w_j \neq u, 1 \leq j \leq q\}. \quad (4)$$

The relation (4) and the fact that M_1 is a model for r″ involve that M_1 is a model for r′, hence for P|M.

A multi-valued stable model for P is defined as a fixed point of the operator Γ^*.

Definition 6: A multi-valued interpretation M for a program P is called a multi-valued stable model for P if M is a fixed point of Γ^*.

Well-founded models: For definition of well-founded models we need to introduce an operator, denoted W, defined on the set of all multi-valued interpretations.

For an interpretation I, if J=W(I) and $V_J = (S_0, S_1,..., S_n)$, we define the sets S_j, $0 \leq j \leq n$.

Definition 7: Let I be an interpretation. We define the sets S_j, $0 \leq j \leq n$ in the following manner:

a. for every j, $1 \leq j \leq n$, a ground atom A is included in S_j iff
a1. for every ground instantiated rule r of P of the form: $r \equiv A \leftarrow L_1,..., L_m$, we have : $\min\{\hat{I}(L_j), \hat{I}(L_j) \neq u, 1 \leq j \leq m\} \leq v_j$ and
a2. there exists a ground instantiated rule r_1 of P with the form: $r_1 \equiv A \leftarrow Q_1,..., Q_h$, such that: $\hat{I}(Q_i) \neq u$, for every i, $1 \leq i \leq h$ and $\min\{\hat{I}(Q_i), 1 \leq i \leq h\} = v_j$.
b. A set of atoms V from H is called unfounded set of P with respect to I if every atom A from V satisfies the following property:
for each ground instantiated rule r of P, having the form: $r \equiv A \leftarrow L_1,..., L_m$, one of the following statements holds:
b1. there is i, $1 \leq i \leq m$, such that $\hat{I}(L_i) = 0$ or
b2. there is i, $1 \leq i \leq m$, such that L_i is an atom and $L_i \in V$.
c. We consider S_0 as the union of all unfounded sets of P with respect to I.

Remark 2: If V_1 and V_2 are unfounded sets of P with respect to I, then their union $V_1 \cup V_2$ is also an unfounded set with respect to I.

Proposition 4: The operator W is monotonic with respect to Fitting ordering $\leq_F$.

Proof. Let I and J be two interpretations, such that $I \leq_F J$. Let $V_I = (S_0, S_1,...,S_n)$ and $V_J = (T_0, T_1,..., T_n)$.

We have $S_j \subseteq T_j$ for every j, $0 \leq j \leq n$. That means: if $\hat{I}(L_i) \neq u$ then $\hat{J}(L_i) \neq u$ and $\hat{J}(L_i) = \hat{I}(L_i)$, for every literal L_i.

If $V_{w(I)} = (S'_0, S'_1,..., S'_n)$ and $V_{w(J)} = (T'_0, T'_1,..., T'_n)$ then it obtains that $S'_j \subseteq T'_j$, for every j, $1 \leq j \leq n$. (1)

The relations $S_0 \subseteq T_0$ and $S_n \subseteq T_n$ imply the following statement: every unfounded set of P with respect to I is an unfounded set of P with respect to J.

We obtain $S'_0 \subseteq T'_0$. This relation and those from (1) involve $W(I) \leq_F W(J)$.

Now, we define a sequence of interpretations using the operator W defined above.

Definition 8: Let α range over countable ordinals. We define recursively the interpretations I_α and I^∞ as follows:

1. For ordinal 0, $I_0 = (\varnothing,..., \varnothing)$, where $\varnothing$ is the empty set;
2. For the limit ordinal $\alpha: I_\alpha = \bigcup_{\beta<\alpha} I_\beta$;
3. For successor ordinal $\alpha = \gamma+1$: $I_\alpha = W(I_\gamma)$;
4. $I^\infty = \bigcup_\alpha I_\alpha$.

Remark 3

i. The interpretation I^∞ is the least fixed point of W with respect to the Fitting ordering $\leq_F$.
ii. There exists a countable ordinal α, such that $I^\infty = I_\alpha$.

Let us denote the interpretation I^∞ by I_P.

Theorem 2: The sequence of interpretation I_α as defined in the Definition 8 is a monotonic sequence of interpretations with respect to $\leq_F$-ordering and moreover it is a sequence of models for P.

Proof: The monotony of the sequence of interpretations results from the Proposition 4.

By the Definition 2, I_0 is a model for P. Since the operator W is monotonic with respect to ordering$\leq_F$, it results by induction on ordinals α the following statement:

for every ground literal L and $\gamma<\alpha$, if $I_\gamma(L) \neq u$ then $I_\alpha(L) \neq u$ and $I_\gamma(L) = I_\alpha(L)$ (1)

Assume that I_γ is a model for P. Let us show that $I_{\gamma+1}$ is also a model for P, where γ is an arbitrary ordinal. Let $r \equiv A \leftarrow L_1,..., L_m$ be a ground instantiated rule of P. If $I_{\gamma+1}(A) = u$, then $I_{\gamma+1}$ satisfies r.

In the case $I_{\gamma+1}(A) \neq u$, let $V_{I_{\gamma+1}} = (S_0, S_1,..., S_n)$. There exists j, $0\leq j\leq n$, such that $A \in S_j$. We have $I_{\gamma+1}(A) = v_j$. Using the Definition 7 and the relation (1), we obtain that $I_{\gamma+1}$ satisfies r.

Now, let A be a limit ordinal. Assume that I_β for every $\beta<\alpha$ are models for P. Let us show that I_α is model.

Let $V_{I_\beta} = (S_0^\beta,\ldots S_n^\beta)$. We have $V_{I_\alpha} = \left(\bigcup_{\beta<\alpha} S_0^\beta,\ldots,\bigcup_{\beta<\alpha} S_n^\beta\right)$

Let r be defined as above. If $I_\alpha(A) = u$, then I_α satisfies r. In the case $I_\alpha(A) \neq u$, there is h, $0\leq h\leq n$, such that $A \in \bigcup_{\beta<\alpha} S_h^\beta$. The sequence of sets S_h^β, $\beta<\alpha$ is ascending monotonic with respect to the inclusion. Let β_1 be the first ordinal such that $A \in S_h^{\beta_1}$. We have $I_{\beta 1}(A)= v_h$ and $I_{\beta 1}$ is a model for r. Since $\beta_1<\alpha$ and using the relation (1), it results that I_α satisfies r.

Stable Semantics versus well-founded semantics: In this section we point out a relation between the stable semantics and the well-founded semantics, namely the well-founded model of P is the least stable model of P with respect to$\leq_F$-ordering.

Theorem 3: Let P be a normal logic program. Then P admits$\leq_F$-least stable model. Moreover, this model coincides with the well-founded model of P.

Proof: Let I_P be the well-founded model for P and λ be the minimum ordinal such that $I_\lambda = I_{\lambda+1}$ (from the Definition 8).

Firstly, we show that I_P is a stable model for P. Let P′ be P/I_P and M_1 be an arbitrary model for P′, such that $M_1 \leq_s I_P$. It must show that $M_1 = I_P$. Let V_{M_1} be the vector $(T_0,..., T_n)$ and $V_{I_\lambda} = (S_0^\lambda,\ldots,S_n^\lambda)$.

The relation $M_1 \leq_s I_P$ is equivalent with:

i. $S_0^\lambda \subseteq T_0$ and
ii. $T_n \subseteq S_n^\lambda$ and
iii. $T_h \subseteq S_h^\lambda \cup\ldots\cup S_{n-1}^\lambda$ for every h, $1\leq h\leq n-1$.

Assume that $M_1 \neq I_P$. Then, we have one of the following assertions:

a. $S_n^\lambda \not\subseteq T_n$ or
b. $S_0^\lambda \subset T_0$ or
c. there is h, $1\leq h\leq n-1$ such that $S_h^\lambda \not\subseteq T_h \cup\ldots\cup T_{n-1}$.

The sign "$\subset$" denotes the strict inclusion and "$\not\subseteq$" means "not included".

In the case a) let us consider α the least ordinal such that $S_n^{a+1} \not\subseteq T_n$. where, $V_{I_\alpha} = (S_0^\alpha,\ldots,S_n^\alpha)$ and I_a is specified in the Definition 8, for every ordinal α. It results that $S_n^\alpha \subseteq T_n$ and there exists a ground atom A, such that $A \in S_n^{a+1}$ and $A \notin T_n$. By the definition of $S_n^{\alpha+1}$, there is a ground instantiated rule r of P, having the form: $r_1 \equiv A \leftarrow B_1... B_m \sim D_1...\sim D_p$,

where, B_j, $1 \leq j\leq m$ and D_l, $1 \leq l\leq p$ are ground atoms with the properties:

$\hat{I}_\alpha(B_j) = 1$ for every j, $1 \leq j\leq m$ and $\hat{I}_\alpha(D_l) = 0$ for every l, $1 \leq l\leq p$.

Let r'_1 be the rule from P' corresponding to r_1. Then $r'_1 \equiv A \leftarrow B_1,...,B_m, c_{v_1},\ldots,c_{v_p}$, where, $v_j = \hat{I}_\lambda(D_j)$ for every j, $1 \le j \le p$. Since $S^\alpha_n \subseteq T_n$, we obtain that $M_1(B_j) = 1$, $j = \overline{1,m}$. Since $I_\alpha \le_F I_\lambda$, it results $I_\lambda(D_j) = 0$, $j = \overline{1,p}$, hence $v_j = 1$, for every j, $1 \le j \le p$. We have: M_1 is a model for r'_1. This implies $M_1(A) = 1$, hence $A \in T_n$, which is impossible. Therefore, we have $T_n = S_n^\lambda$.

In the case c)let α be the least ordinal, such that $S_h^{\alpha+1} \not\subset T_h \cup \ldots \cup T_{n-1}$ (1)

It results: $S_h^\alpha \not\subset T_h \cup \ldots \cup T_{n-1}$ (2)

Using the relation (1), we obtain: there is A, such that $A \in S_h^{\alpha+1}$ and $A \notin T_h \cup ... \cup T_{n-1}$ (3)

$A \in S_h^{\alpha+1}$ implies: for every $r \in M_A$, $r \equiv A \leftarrow L_1,..., L_m$, we have $\hat{I}_\alpha(body(r)) \le v_h$ (4)

and there is $r_1 \in M_A$, $r_1 \equiv A \leftarrow Z_1,..., Z_p$, such that $\hat{I}_\alpha(Z_j) \ne u$, for every j, $1 \le j \le p$ and $\min \{\hat{I}_\alpha(Z_j)\} = v_h$. (5)

Let r_1 from (5) be expressed as follows: $r_1 \equiv A \leftarrow B_1... B_m \sim D_1...\sim D_q$

We have: $\hat{I}_\alpha(B_i) \ne u$, $i = \overline{1,m}$ and $\hat{I}_\alpha(\sim D_j) \ne u$, $i = \overline{1,q}$, which imply: $\hat{I}_\alpha(B_i) \ge v_h$ and $\hat{I}_\alpha(\sim D_j) \ge v_h$ $i = \overline{1,q}$. (6)

Let r'_1 be the clause from P/I_P corresponding to r_1: $r'_1 \equiv A \leftarrow B_1,...,B_m, c_{v_1},\cdots c_{v_q}$ where, $v_j = \hat{I}_P(D_j)$, $j = \overline{1,q}$. Since $I_\alpha \le_F I_P$, we have $\hat{I}_P(\sim D_j) = \hat{I}_\alpha(\sim D_j)$, for every j, $j = \overline{1,q}$, hence $v_j \ge v_h$ for $j = \overline{1,q}$. (7)

We have $I_\alpha(B_i) > 0$, $i = \overline{1,m}$. If $I_\alpha(B_i) = 1$, then $I_\lambda(B_i) = 1$ and using $T_n = S_n^\lambda$, it obtains that $M_1(B_i) = 1$. If $I_\alpha(B_i) < 1$, then using (2) it results: $B_j \in T_h \cup ... \cup T_{n-1}$, hence $M_1(B_i) \ge v_h$. Since M_1 satisfies r'_1, we have $M_1(A) \ge v_h$. We show that $M_1(A) \ne 1$. Assume the contrary: $M_1(A) = 1$. Using $T_n = S_n^\lambda$, we obtain $A \in S_n^\lambda$. (8)

From $A \in S_h^{\alpha+1}$, it results $A \in S_h^\lambda$, with h<n. (9)

But $S_h^\lambda \cap S_n^\lambda = \varnothing$ for h<n. The relations (8) and (9) constitute a contradiction.

From $M_1(A) \ge v_h$ and $M_1(A) < 1$ we obtain $A \in T_h \cup ... \cup T_{n-1}$ which contradicts the relation (3).
In conclusion for the case c), we have:
$S_h^\lambda \subseteq T_h \cup ... \cup T_{n-1}$ for every $h = \overline{1,n-1}$.
Using (iii), it results $S_h^\lambda = T_h$, $h = \overline{1,n-1}$.

In the case b), namely $S_0^\lambda \subset T_0$, we show that $T_0 \subseteq S_0^\lambda$, which will be a contradiction.

Let A be from T_0, hence $M_1(A) = 0$. Let r be a ground instantiated rule from P, having the form: $r \equiv A \leftarrow B_1... B_m \sim D_1...\sim D_p$
The clause corresponding to r from P/I_P is r':

$r' \equiv A \leftarrow B_1,...,B_m, c_{v_1},\cdots c_{v_q}$ where, $v_j = \hat{I}_\lambda(D_j)$, $j = \overline{1,p}$.

Since M_1 is a model for r', it follows that there exists i, $1 \le i \le m$ such that $M_1(B_i) = 0$ or there is j, $1 \le j \le p$, such that $c_{v_j} = 0$

For every $c_{v_j} = 0$ we have $\hat{I}_\lambda(\sim D_j) = 0$. (10)

If $c_{v_j} > 0$ for all j, $1 \le j \le p$, then there is i, $1 \le i \le m$, such that $M_1(B_i) = 0$,
hence $B_i \in T_0$. (11)

The assertions (10) and (11) say that T_0 is an unfounded set with respect to I_λ.

If $V_{W(I_\lambda)} = (T'_0,\ldots T'_n)$, then $T_0 \subseteq T'_0$.

But $W(I_\lambda) = I_\lambda$, hence we have $T'_0 = S^\lambda_0$, which implies $T_0 \subseteq S^\lambda_0$ therefore a contradiction.

Thus, we have $S^\lambda_0 = T_0$, hence $M_1 = I_P$ and $I_\lambda = I_P$ is a stable model for P.

Secondly, we show that I_P is $\le_F$-least stable model for P.

Let M_1 be a stable model for P. Let V_{M1} be defined as follows:

$$V_{M1} = (T_0, T_1,..., T_n).$$

The model M_1 is the least model of P/M_1 with respect to the ordering $\le_s$. Let I_a be the interpretations as in the

Definition 8. Let $V_{I_\alpha} = (S_0^\alpha,\ldots,S_n^\alpha)$.

We show by induction on α the following relations: $S^\alpha \subseteq T_k$, for every k, $0 \le k \le n$. (12)

Since $I_0 = (\varnothing,..., \varnothing)$, we have that (12) are true for k = 0.
Assume that (12) is true for every ordinal α, $\alpha < ß$.
If ß is limit ordinal, then (12) is true for ß.
Now let ß be a successor ordinal, $ß = \alpha + 1$.
It must show that $S_k^{\alpha+1} \subseteq T_k$, $k = \overline{0,n}$. (13)
Let us distinguish two cases:
1) $k \ge 1$, 2) $k = 0$.

In case 1) let A be from $S^{\alpha+1}_k$. We have: for every $r \in M_A$, having the form: $r \equiv A \leftarrow Z_1... Z_q$, $\min \{\hat{I}_\alpha(Z_j), \hat{I}_\alpha(Z_j) \ne u\} \le v_k$ and there is $r_1 \equiv A \leftarrow S_1... S_p$, such that $\hat{I}_\alpha(S_j) \ne u$, for every j, $1 \le j \le p$ and $\min \{\hat{I}_\alpha(S_j), 1 \le j \le p\} = v_k$. From (12) it results:

$\hat{I}_\alpha(L) = M_1(L)$ for every ground literal L. (14)

Since M_1 is also a model for P, we have $M_1(A) \neq u$ and moreover $M_1(A) \geq v_k$. Let us denote $M_1(A) = v_h$.

If we assume that $v_h > v_k$, then we define an interpretation M_1' as follows:

$M_1'(B) = M_1(B)$ if $B \neq A$ and v_k otherwise.

It results that M_1' is a model for P/M_1, $M_1' \leq_s M_1$ and $M_1' \neq M_1$, which contradicts the fact M_1 is the least model for P/M_1 with respect to the ordering $\leq_S$. Hence, we have $v_h = v_k$, i.e. $A \in T_k$.
In the case 2) let A be form $S_0^{\alpha+1}$.

In this case, for every $r \in M_A$, of the form: $r \equiv A \leftarrow L_1 \ldots L_m$, there is i such that $\hat{I}_\alpha(L_i) = 0$, or there is i, such that L_i is atom and $L_i \in S_0^{\alpha+1}$. (15)

Using the hypothesis of induction (12), we obtain: $\hat{I}_\alpha(L_i) = 0$ implies $M_1(L_i) = 0$.
Let $r \in M_A$ be of the form $r \equiv A \leftarrow B_1 \ldots B_q \sim D_1 \ldots \sim D_p$
The clause corresponding to r from P/M_1 is r′:
$r' \equiv A \leftarrow B_1, \ldots, B_q, c_{v_1}, \ldots c_{v_p}$ where $v_j = \hat{M}_1(D_j)$, $j = \overline{1,p}$.
If $M_1(A) \neq u$ and $M_1(A) = 0$, then $A \in T_0$.

If $M_1(A) \neq u$ and $M_1(A) = v_k$ with $v_k \geq 0$, then we consider a model M_1' defined by following:

$$T'_0 = T_0 \cup S_0^{\alpha+1}, T'_j = T_j - S_0^{\alpha+1} j = \overline{1,n}$$

and $V_{M'_1} = (T'_0, \ldots, T'_n)$

Since $S_0^{\alpha+1}$ is an unfounded set with respect to M_1, we have M_1' is a model for P/M_1.

Moreover, since $M_1' \leq_s M_1$ and $M_1' \neq M_1$, it results a contradiction.

If $M_1(A) = u$, we consider the same interpretation M_1' as it was described above, which implies a contradiction. It results the statement (13). Taking in (13) $\alpha = \lambda$, it obtains that $I_P \leq_F M_1$, therefore I_P is the $\leq_F$-least stable model for P.

CONCLUSION

This study introduced new semantics for general logic programs considering a set of n+1-truth logic values and an undefined value. One of semantics is of type well-founded and the other is of type stable. We have studied a relationship between the two semantics. For n=1 and u=1/2, the results of Przymusinski[4] are obtained.

REFERENCES

1. Van Gelder, A., K.A. Ross and J.S. Schlipf, 1991. The well-founded semantics for general logic programs. JACM., 38: 620-650.
2. Gelfond, M. and V. Lipschitz, 1988. The stable model semantics for logic programming. In: Kowalski, R. and K. Bowen, (Eds.), Proceedings of the Fifth Logic programming Symposium, MIT Press, Cambridge, MA., pp: 1070-1080.
3. Bidoit, N. and C. Froidevaux, 2001. General Logical Databases and Programs: Default Logic Semantics and Stratification, John Wiley and Sons.
4. Przymusinski, T., 1990. Well-founded semantics coincides with three-valued stable semantics. Fundamenta Informaticae, 13: 445-463.
5. Luong, V.P., 1999. Between well-founded semantics and stable model semantics, 1999. International Database Engineering and Applications Symposium. Aug., 02-04, pp: 270. www.computer.org/proceedings/ideas/0265/02650270abs.htm.
6. Fitting, M.C., 1991. Well-founded semantics, generalized. In: Saraswat, V. and K. Ueda, (Eds.), Logic programming. Proceedings of the 1991 International Symposium, MIT Press, Cambridge, M.A., pp: 71-84.
7. Przymusinski, T., 1990. Extended stable semantics for normal and disjunctive programs. In: Warren, D.H.D. and P. Szeredi, (Eds.), Proc. Seventh Int. Conf. Logic Programming, MIT Press, Cambridge, MA., pp: 459-477.
8. Loyer, Y. and S. Umberto. The well-founded semantics in normal logic programs with uncertainty. http://faure.iei.pi.cnr.it/~straccia/download/papers/FLOPS02/FLOPS02.pdf
9. Malfon, B., 1994. Characterization of some semantics for logic programs with negation and application to program validation. In Maurice, B., (Ed.), Intl. Logic Programming Symp., MIT Press, pp: 91-105.
10. Lallouet, A., 1996. Une notion de composant logiciel en programation logique. JFPLC'96, pp: 83-94.