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A Class of Explicit Fourth Order Method with Phase Lag of Order Six for Second Order Initial Value Problems

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Abstract: We derive the explicit fourth order two-step methods for the numerical integration of second order initial value problems $\ddot{y} = f(t, y), y(t_0) = \eta_0, \dot{y}(t_0) = \dot{\eta}_0$. Then we discuss the P-stability and phase lag of the methods. Next we discuss particular methods and present the numerical result of these methods.

Key words: Explicit fourth order method, numerical integration, P-stability, phase lag, initial value problem

INTRODUCTION

A class of two-step hybrid methods for the numerical integration of second order initial value problem

$$\ddot{y} = f(t, y), y(t_0) = \eta_0, \dot{y}(t_0) = \dot{\eta}_0. \quad (1)$$

is defined in the following. Let h>0 denote the stepsize $t_n = t_0+nh$, h = 0,1,2,... and set $y_n = y(t_n)$, $f_n = f(t_n,y_n)$.

We consider the method of the form

$$y_{n+1} = 2y_n - y_{n-1} + h^2 m_1 \quad (2)$$

$$y_{n\pm\alpha_1} = A_\pm \tilde{y}_{n+1} + B_\pm y_n + C_\pm y_{n-1} + h^2 m_2 \quad (3)$$

$$\tilde{y}_{n+1} = 2y_n - y_{n-1} + h^2 \ddot{y}_n \quad (4)$$

$$\ddot{y}_{n\pm\alpha 1} = f(t_{n\pm\alpha 1}, y_{n\pm\alpha 1}),\ \tilde{f}_{n+1} = f(t_{n+1}, \tilde{y}_{n+1}) \quad (5)$$

where:

$$m_1 = \beta_0 \tilde{\ddot{y}}_{n+1} + \beta_0 \ddot{y}_{n-1} + \beta_1 (\ddot{y}_{n+\alpha 1} + \ddot{y}_{n-\alpha 1}) + \gamma \ddot{y}_n$$

and $m_2 = s_\pm \tilde{\ddot{y}}_{n+1} + q_\pm \ddot{y}_{n-1} + u_\pm \ddot{y}_{n-1}$

The methods are fourth order accurate if satisfy the following conditions.

i) $1-\gamma-2\beta_0-2\beta_1 = 0,$

ii) $\beta_1 n_1 = 0,$

iii) $\frac{1}{12} - \beta_0 - \beta_1 \alpha_1^2 = 0,$

iv) $\frac{1}{6} - 2\beta_0 - \beta_1 \alpha_1 n_2 = 0,$

v) $\frac{1}{12} - \beta_0 - \beta_1 n_3 = 0,$

vi) $\frac{1}{12} - \beta_0 - \frac{1}{2}\beta_1 n_4 = 0,$

vii) $\beta_1 \alpha_1^2 n_1 = 0,$

viii) $\beta_1 \alpha_1 n_5 = 0,$

ix) $\beta_1 \alpha_1 n_6 = 0,$

x) $\beta_1 n_7 = 0,$

xi) $\beta_1 n_8 n_{10} + \beta_1 n_9 n_{11} = 0,$

xii) $\beta_1 \left[n_8^3 + n_9^3\right] = 0.$

xiii) $A_\pm + B_\pm + C_\pm = 1$

where:

$n_1 = n_8+n_9$, $n_2 = n_8-n_9$, $n_3 = n_{10}+n_{11}$, $n_4 = n_8^2+n_9^2$, $n_5 = n_{10}-n_{11}$, $n_6 = n_8^2-n_9^2$, $n_7 = n_{12}-n_{13}$, $n_8 = A_+-C_+$, $n_9 = A_--C_-$,

$n_{10} = \frac{A_+ + C_+}{2} + s_+ + q_+ + u_+$ and $n_{11} = \frac{A_- + C_-}{2} + s_- + q_- + u_-$,

$n_{12} = 1/6(A_+-C_+)+s_+-u_+$, $n_{13} = \frac{1}{6}(A_--C_-) + s_- - u_-$

and $n_{14} = s_+ + s + u_+ + u_-$ and the local truncation error is

$$\text{LTE} = \lambda_1 h^6 + O(h^7) \quad (6)$$

$$\lambda_1 = \left(\frac{1}{360} - \frac{1}{12}\beta_0 - \frac{1}{12}\beta_1 \alpha_1^4\right)\frac{\partial^4 f}{\partial t^4}$$

$$+\left[\frac{1}{90}-\frac{1}{3}\beta_0-\frac{1}{6}\beta_1\alpha_1^3 n_2\right]\sum_j \frac{\partial^4 f}{\partial t^3 \partial y_j} y'$$

$$+\left\{\frac{1}{60}-\frac{1}{2}\beta_0-\frac{1}{2}\beta_1\alpha_1^2 n_3\right\}\sum_j \frac{\partial^3 f}{\partial t^2 \partial y_j}\ddot{y}$$

$$+\left\{\frac{1}{60}-\frac{1}{2}\beta_0-\frac{1}{4}\beta_1\alpha_1^2 n_4\right\}\sum_{j1}\sum_{j2} \frac{\partial^4 f}{\partial t^2 \partial y_{j1}\partial_{j2}} y'_{j1}y'_{j2}$$

$$+\left\{\frac{1}{90}-\frac{1}{3}\beta_0-\beta_1\alpha_1 n_7\right\}\sum_j \frac{\partial^2 f}{\partial t \partial y_j} y^{(3)}$$

$$+\left\{\frac{1}{30}-\beta_0-\beta_1\alpha_1\left(n_8 n_{10}-n_9 n_{11}\right)\right\}\sum_{j1}\sum_{j2} \frac{\partial^3 f}{\partial t \partial y_{j1}\partial y_{j2}} y'_{j1}\ddot{y}_{j2}$$

$$+m_1\sum_{j1}\sum_{j2}\sum_{j3} \frac{\partial^4 f}{\partial t \partial y_{j1}\partial_{j2}\partial_{j3}} y'_{j1}y'_{j2}y'_{j3}$$

$$+m_2\sum_j \frac{\partial f}{\partial y_j} y^{(4)}$$

$$+m_3\sum_{j1}\sum_{j2} \frac{\partial^2 f}{\partial y_{j1}\partial y_{j2}} \ddot{y}_{j1}\ddot{y}_{j2}$$

$$+m_4\sum_{j1}\sum_{j2} \frac{\partial^2 f}{\partial y_{j1}\partial y_{j2}} y'_{j1}y^{(3)}_{j2}$$

$$+m_5\sum_{j1}\sum_{j2}\sum_{j3} \frac{\partial^3 f}{\partial y_{j1}\partial y_{j2}\partial y_{j3}} y'_{j1}y'_{j2}\ddot{y}_{j3}$$

$$+m_6\sum_{j1}\sum_{j2}\sum_{j3}\sum_{j4} \frac{\partial^4 f}{\partial y_{j1}\partial y_{j2}\partial y_{j3}\partial y_{j4}} y'_{j1}y'_{j2}y'_{j3}y'_{j4}$$

where,

$$m_1 = \frac{1}{90}-\frac{1}{3}\beta_0-\frac{1}{6}\beta_1\alpha_1\left(n_8^3+n_9^3\right),$$

$$m_2 = \frac{1}{360}+\frac{1}{24}\beta_1\left(n_8+n_9\right)-\frac{1}{2}\beta_1 n_{14},$$

$$m_3 = \frac{1}{120}-\frac{1}{4}\beta_0-\frac{1}{2}\beta_1 n_{10}^2-\frac{1}{2}\beta_1 n_{11}^2,$$

$$m_4 = \frac{1}{90}-\frac{1}{3}\beta_0-\beta_1\left[n_8 n_{12}+n_9 n_{13}\right],$$

$$m_5 = \frac{1}{60}-\frac{1}{2}\beta_0-\frac{1}{2}\beta_1\left(n_8^2 n_{12}+n_9^2 n_{13}\right),$$

and

$$m_6 = \frac{1}{360}-\frac{1}{12}\beta_0-\frac{1}{24}\beta_1\left(n_8^4+n_9^4\right).$$

Further from condition (ii) either $\beta_1 = 0$ or $A_+ + A_- - C_+ - C_- = 0$. Also from condition (vii) either $\beta_1 = 0$ or $\alpha_1 = 0$ or $A_+ + A_- - C_+ - C_- = 0$. We consider the case where $\beta_1 = 0$ and $\beta_1 \neq 0$.

Case 1: If $\beta_1 = 0$ then from condition (iii) $\beta_0 = 1/12$ and from condition (i) $\gamma = 5/6$. Then the local truncation error from (6) becomes

$$\text{LTE} = -\frac{1}{720}\left[3y^{(6)} - 5\sum_j \frac{\partial f}{\partial y_j} y_j^{(4)}\right]h^6 + O(h^7). \qquad (7)$$

Case 2: If $\beta_1 \neq 0$ then either $\alpha_1 = 0$ or $\alpha_1 \neq 0$.

(a) If $\alpha_1 = 0$ then we have from the conditions $\alpha_1 = 0$, $s_+ + s_- = u_+ + u_-$, $\beta_0 = 1/12$, $\gamma = 5/6 - 2\beta_1$, $C_\pm = A_\pm$, $B_\pm = 1 - 2A_\pm$, $q_+ + q_- = -(A_+ + A_-) - 2(u_+ + u_-)$, (8)

and the LTE becomes

$$\text{LTE} = \left\{-\frac{1}{240}y^{(6)} + m_7 - m_8\right\}h^6 + O(h^7) \qquad (9)$$

where, $m_7 = \left[\frac{5}{720} - \beta_1\left(s_+ + s_-\right)\right]\sum_j \frac{\partial f}{\partial y_j} y^{(4)}$

and $m_8 = \beta_1\left(A_+ + s_+ + q_+ + u_+\right)^2\sum_{j1}\sum_{j2} \frac{\partial^2 f}{\partial y_{j1}\partial y_{j2}}\ddot{y}_{j1}\ddot{y}_{j2}$.

Observe that Chawla and Rao[1] method is of this class. They choose the parameters as $\alpha_1 = 0$, $A_+ = A_- = C_+ = C_- = 0$, $B_+ = B_- = 1$, $q_+ = q_- = 2\alpha$, $u_+ = u_- = s_+ = s_- = -\alpha$, $\beta_0 = 1/12$, $\beta_1 = 5/12$, $\gamma = 0$ and $\alpha = -1/300$ (10)
and

LTE

$$= -\frac{1}{720}\left[3y_{n+1}^{(6)} - 5y_{n+1}^{(4)}(f_y)_{n+2} - 600\alpha y_{n+1}^{(4)}(f_y)_{n+1}\right]h^6 + O(h^8) \qquad (11)$$

(b) If $\alpha_1 \neq 0$ then fourth order conditions becomes

$\beta_0 = \frac{1}{12} - \beta_1\alpha_1^2$, $\gamma = \frac{5}{6} + 2\beta_1\left(\alpha_1^2 - 1\right)$, $s_- = u_+ + u_- - s_+$, $B_+ = 1 + \alpha_1 - 2A_+$, $B_- = 1 - \alpha_1 - 2A_-$, $C_+ = A_+ - \alpha_1$, $C_- = A_- + \alpha_1$,

$q_+ = \frac{1}{2}\left(\alpha_1^2 + \alpha_1\right) - A_+ - s_+ - u_+$, $q_- = \frac{1}{2}\left(\alpha_1^2 - \alpha_1\right) - A_- - s_- - u_-$

and the local truncation error becomes

$$\text{LTE} = \lambda_2 h^6 + O(h^7) \qquad (12)$$

where

$$\lambda_2 = \frac{1}{12}\left(\frac{1}{30} - \beta_0 - \beta_1\alpha_1^4\right)y^{(6)} + 2\beta_1\alpha_1\left[\frac{1}{6}\alpha_1\left(\alpha_1^2 - 1\right) - \left(s_+ - u_+\right)\right]$$
$$\times\left[\sum_j \frac{\partial^2 f}{\partial t \partial y_j} y^{(3)} + \sum_{j1}\sum_{j2} \frac{\partial^2 f}{\partial y_{j1}\partial y_{j2}} y'_{j1}y^{(3)}_{j2}\right]$$
$$+\left[\frac{1}{12}\left(\beta_0 + \beta_1\alpha_1^4\right) - \beta_1\left(s_+ + s_-\right)\right]\sum_j \frac{\partial f}{\partial y_j} y^{(4)}$$

P-Stability: Lambert and Watson[2] have given the definition of P-stability

To analyses the stability properties of the method, we apply the method (2), (3), (4) to the scalar test equation:

$$\ddot{y} + c^2 y = 0, \qquad c \text{ real.} \tag{13}$$

We obtain the stability polynomial

$$p(r) = r^2 - \left\{2 - K_1H^2 + K_2H^4 - K_3H^6\right\}r + \left\{1 + \beta_1 K_4 H^2 + \beta_1 K_5 H^4\right\} \tag{14}$$

where:

$K_1 = [2\beta_0 + \gamma + \beta_1\,(2A_+ + 2A_- + B_+ + B_-)]$,

$K_2 = [\beta_0 + \beta_1\,(A_+ + A_- + 2s_+ + 2s_- + q_+ + q_-)]$,

$K_3 = \beta_1\,(s_+ + s_-)$, $K_4 = (C_+ + C_- - A_+ - A_-)$,

$K_5 = (s_+ + s_- - u_+ - u_-)$ and $H = ch$

Equation (14) is a difference equation of the form[3]

$$\tau_2(H)y_{n+1} + \tau_1(H)y_n + \tau_0(H)y_{n-1} = 0 \tag{15}$$

where:

$\tau_2 = \tau_0 = 1$ and $\tau_1(H) = -2 + H^2 - \frac{1}{12}H^4 + \beta_1\,(s_+ + s_-)H^6$.

When we apply the fourth order condition then the stability polynomial is of the form $p(r) = r^2 - 2B(H)r + 1$,

where, $B(H) = 1 - \frac{1}{2}H^2 + \frac{1}{24}H^4 - \frac{1}{2}\beta_1\,(s_+ + s_-)H^6$.

Making the Routh-Herwitz transformation $r = \frac{1+z}{1-z}$; (15) becomes

$(\tau_2 - \tau_1 + \tau_0)z^2 + 2(\tau_2 - \tau_0)z + (\tau_2 + \tau_1 + \tau_0) = 0$.

Thus, the necessary and sufficient conditions for P-stability are $(\tau_2 - \tau_1 + \tau_0) \geq 0$, $\tau_2 - \tau_0 = 0$, $(\tau_2 + \tau_1 + \tau_0) \geq 0$.

Since $\tau_2 = \tau_0$, then the condition becomes

$(2\tau_2 - \tau_1)z^2 + (2\tau_2 + \tau_1) = 0$ and the necessary and sufficient conditions for P-stability are $2\tau_2 - \tau_1 \geq 0$ and

$$2\tau_2 + \tau_1 \geq 0 \tag{16}$$

where:

$$2\tau_2 - \tau_1 = 4 - H^2 + \frac{1}{12}H^4 - \beta_1\,(s_+ + s_-)H^6 \geq 0 \tag{17}$$

and

$$2\tau_2 + \tau_1 = H^2\left[1 - \frac{1}{12}H^2 + \beta_1\,(s_+ + s_-)H^4\right] \geq 0 \tag{18}$$

Condition (18) is satisfied iff $\frac{1}{144} - 4\beta_1\,(s_+ + s_-) \leq 0$ $\beta_1\,(s_+ + s_-) \geq \frac{1}{576}$. Thus the methods are P-stable if satisfy the condition (17) and (18) with $\beta_1\,(s_+ + s_-) \geq \frac{1}{576}$.

Phase LAG: When the method (2), (3) and (4) is applied to the scalar equation (13), we have the recurrence relation[4]:

$$y_{n+1} - \left[2 - H^2 + \frac{1}{12}H^4 - \beta_1\,(s_+ + s_-)H^6\right]y_n + y_{n-1} = 0$$

Substituting $y_n = e^{n\theta H}$ we have

$$e^{2\theta H} - \left[2 - H^2 + \frac{1}{12}H^4 - \beta_1\,(s_+ + s_-)H^6\right]e^{\theta H} + 1 = 0$$

on expansion of $e^{2\theta H}$ and $e^{\theta H}$, we have the form

$$\begin{aligned} 0 &= (1+\theta^2)H^2 + (\theta^3 + \theta)H^3 \\ &+ \left(\frac{7}{12}\theta^4 + \frac{1}{2}\theta^2 - \frac{1}{12}\right)H^4 \\ &+ \left(\frac{1}{4}\theta^5 + \frac{1}{6}\theta^3 - \frac{1}{12}\theta\right)H^5 \\ &+ \left[\frac{31}{360}\theta^6 + \frac{1}{24}\theta^4 - \frac{1}{24}\theta^2 + \beta_1\,(s_+ + s_-)\right]H^6 \\ &+ \left[\frac{1}{40}\theta^7 + \frac{1}{120}\theta^5 - \frac{1}{72}\theta^3 + \beta_1\,(s_+ + s_-)\,\theta\right]H^7 \\ &+ \left[\frac{127}{20160}\theta^8 + \frac{1}{720}\theta^6 - \frac{1}{288}\theta^4 + \frac{1}{2}\beta_1\,(s_+ + s_-)\,\theta^2\right]H^8 \end{aligned} \tag{19}$$

Let $\theta = \eta_0 + \eta_1 H + \eta_2 H^2 + \eta_3 H^3 + \eta_4 H^4$. Substitutes in (19) and then comparing the coefficients of H^j, j = 2, 3,4,5,6 to zero, we get

$$\eta_0 = i,\ \eta_1 = 0,\ \eta_2 = 0,\ \eta_3 = 0,\ \eta_4 = -\frac{\iota}{2}\left[\frac{1}{360} - \beta_1(s_+ = s_-)\right],$$

Thus we have

$$\theta = i + \frac{i}{2}\left[\frac{1}{360} - \beta_1 (s_+ + s_-)\right]H^4 + O(H^5).$$

If $\beta_1 (s_+ + s_-) = \frac{1}{360}$ which is greater than 1/576 (P-stability condition), we may write $\theta = i + \eta_5 H^5 + \eta_6 H^6 \ldots$ and on substituting in (19) and then comparing the coefficients of H^7 and H^8 to zero, we get $\eta_5 = 0$ and $\eta_6 = \frac{i}{40320}$. Thus $\theta = i + \frac{i}{40320} H^6 + O(H^8)$.

Thus the quantity b-1 in the definition of phase lag is

$$b-1 = \frac{i}{40320} H^6 + O(H^8). \quad (20)$$

PARTICULAR METHODS

Case 1: $\beta_1 \neq 0$ and $\alpha_1 = 0$ Observe that Chawla and Rao[1] method is of this class with parameters.

$$\alpha_1 = 0, A_+ = A_- = C_+ = C_- = 0, B_+ = B_- = 1,$$
$$q_+ = q_- = -\frac{1}{150}, u_+ = u_- = s_+ = s_- = \frac{1}{300},$$
$$\beta_0 = \frac{1}{12}, \beta_1 = \frac{5}{12}, \gamma = 0. \quad (21)$$

In this case $\beta_1 (s_+ + s_-) = \frac{1}{360} > \frac{1}{576}$ satisfy the P-stability condition and also phase Lag given by (20). We choose the parameters in two different ways

(a) The points $(t_{n-\alpha 1}, y_{n-\alpha 1})$ and $(t_{n+\alpha 1}, y_{n+\alpha 1})$ are coincident and

(b) $\ddot{y}_{n-\alpha 1} \equiv \ddot{y}_n$.

(a) If the points $(t_{n-\alpha 1}, y_{n-\alpha 1})$ and $(t_{n+\alpha 1}, y_{n+\alpha 1})$ are coincident then

$$A_+ = A_-, B_+ = B_-, C_+ = C_-, s_+ = s_-, q_+ = q_-, u_+ = u_- \quad (22)$$

then from (8)

$$C_+ = C_- = A_+ = A_-, B_\pm = 1\text{-}2A_\pm, u_+ = s_- = u_- = s_+,$$
$$q_+ = q_- = -A_+ - 2s_+, \ \beta_0 = \frac{1}{12}, \gamma = \frac{5}{6} - 2\beta_1 \quad (23)$$

for P-stability we have $\beta_1 (s_+ + s_-) > \frac{1}{576}$ and if we have $\beta_1 (s_+ + s_-) = \frac{1}{360}$ then the method has phase lag given by (20). Thus we take $\beta_1 (s_+ + s_-) = \frac{1}{360} > \frac{1}{576}$ for P-stable method. We choose $\beta_1 = 1$, then $s_+ = 1/720$. Also let $A_+ = 1/2$.

(b) For $\ddot{y}_{n-\alpha 1} \equiv \ddot{y}_n$ we have

$$\alpha_1 = 0, A_- = 0, B_- = 1, C_- = 0, s_- = 0, q_- = 0, u_- = 0. \quad (24)$$

then from (8)

$$s_+ = u_+, \beta_0 = \frac{1}{12}, \gamma = \frac{5}{6} - 2\beta_1, B_+ = 1\text{-}2A_+, C_+ = A_+, \quad (25)$$

and for P-stable with phase lag given by (20), we must have $\beta_1 (s_+ + s_-) = \frac{1}{360}$. Thus we choose $\beta_1 = 1$, then $s_+ = 1/360$. Also let $A_+ = 1/2$

Also we test the method with $A_+ = 0$ and $\beta_1 = 1/3$, then $s_+ = 1/120$. The local truncation error for all these method is

$$LTE = \frac{1}{240}\left[-y^{(6)} + \sum \frac{\partial f}{\partial y} y^{(4)}\right] h^6 + O(h^7) \quad (26)$$

Case 2: If $\beta_1 \neq 0$ and $\alpha_1 \neq 0$, we have chosen the method for $\alpha_1 = 1$. In this case $\ddot{y}_{n-\alpha 1} \equiv \ddot{y}_{n-1}$. Then

$$A\_ = 0, B\_ = 0, C\_ = 1, s\_ = 0, q\_ = 0, u\_ = 0. \quad (27)$$

then from (11), we have

$$u_+ = s_+, \beta_0 = 1/12 - \beta_1, \gamma = 5/6, B_+ = 2\text{-}2A_+,$$
$$C_+ = A_+ - 1, q_+ = 1 - A_+ - 2s_+. \quad (28)$$

For P-stable method with phase lag of order eight given by (20), we must have $\beta_1 (s_+ + s_-) = \frac{1}{360}$ or in this case, $\beta_1 s_+ = \frac{1}{360}$. We choose $\beta_1 = 1$ and $A_+ = 1/2$, then, $s_+ = \frac{1}{360}$. For these methods, LTE is given by (26).

Methods for different examples were tested. The methods are given below.

The methods derived above and used for companion purpose in next section.

M 1: Given by equations (22) and (23) with parameters $\beta_1 = 1$, $S_+ = 1/720$ and $A_+ = 1/2$
M 2: Given by equations (24) and (25) with parameters $\beta_1 = 1$, $S_+ = 1/360$ and $A_+ = 1/2$
M 3: Given by equations (24) and (25) with parameters $\beta_1 = 1/3$, $S_+ = 1/120$ and $A_+ = 0$
M 4: Given by equations (27) and (28) with parameters $\beta_1 = 1$, $S_+ = 1/360$ and $A_+ = 1/2$
M 5: Given by equations (21).

NUMERICAL ILLUSTRATION

We have tried a number of explicit scalar (nonstiff) test problems of the form (1). They give similar results and so we restrict our attention to one oscillatory example.

Example 1: $\ddot{y} + \sinh y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$.

This is a pure oscillation problem whose solution has maximum amplitude unity and period approximately six. We have calculated error as |Error at t = 6|
In Table 1-3 we present the following statistics:

1) Number of evaluation of the differential equation right hand side f, FCN;
2) Number of steps overall, NOST;
3) Number of successful steps to complete the integration, NSST;
4) Number of steps where the stepsize is changes, NCST;
5) Number of failed steps, NFST;
6) Number of steps on which the iteration diverged, NDIV;
7) Number of steps where the stepsize is halved, NHST;

Table 1: Comparison of the method for example 1 with Tol = 10^{-2}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	1.641×10^{-3}	55	18	13	5	6	1	1
M 2	1.650×10^{-3}	55	18	13	5	6	1	1
M 3	3.434×10^{-3}	55	18	13	5	6	1	1
M 4	1.832×10^{-3}	55	18	13	5	6	1	1
M 5	1.654×10^{-3}	55	18	13	5	6	1	1

Table 2: Comparison of the method for example 1 with Tol = 10^{-4}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	2.980×10^{-4}	139	46	35	11	12	3	3
M 2	2.980×10^{-4}	139	46	35	11	12	3	3
M 3	2.959×10^{-4}	139	46	35	11	12	3	3
M 4	2.979×10^{-4}	139	46	35	11	12	3	3
M 5	2.980×10^{-4}	139	46	35	11	12	3	3

Table 3: Comparison of the method for example 1 with with Tol = 10^{-6}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	2.050×10^{-5}	367	122	108	14	13	5	5
M 2	2.050×10^{-5}	367	122	108	14	13	5	5
M 3	2.044×10^{-5}	367	122	108	14	13	5	5
M 4	2.050×10^{-5}	367	122	108	14	13	5	5
M 5	2.050×10^{-5}	367	122	108	14	13	5	5

Next we consider example for system, where we use a test problem a moderately stiff system of two equations

Example 2: $\ddot{y} + \sinh(y_1 + y_2) = 0$, $y_1(0) = 1$, $\dot{y}_1(0) = 0$, $\ddot{y}_2 + 10^4 y_2 = 0$. $y_2(0) = 10^{-8}$, $\dot{y}_2(0) = 0$,

For this example we have deliberately introduced coupling from the stiff (linear) equation to the nonstiff (nonlinear) equation. For this example again we have calculated error as $\|\text{Error at } t = 6\|_\infty$.

Table 4: Comparison of the method for example 2 with with Tol = 10^{-2}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	9.832×10^{-3}	1502	501	480	21	32	3	3
M 2	9.894×10^{-3}	1505	502	480	22	34	3	3
M 3	4.059×10^{-2}	1475	493	466	27	38	4	4
M 4	9.904×10^{-3}	1517	506	484	22	33	3	3
M 5	9.730×10^{-3}	1505	502	483	19	28	3	3

Table 5: Comparison of the method for example 2 with with Tol = 10^{-4}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	4.816×10^{-5}	1024	341	317	24	34	3	3
M 2	3.538×10^{-3}	1756	585	558	27	43	3	3
M 3	1.727×10^{-3}	1357	452	429	23	32	3	3
M 4	5.807×10^{-5}	982	327	312	15	19	3	3
M 5	5.121×10^{-5}	904	301	290	11	13	3	3

Table 6: Comparison of the method for example 2 with with Tol = 10^{-6}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	3.615×10^{-6}	994	331	311	20	23	5	5
M 2	3.615×10^{-6}	994	331	311	20	23	5	5
M 3	1.344×10^{-6}	1300	433	421	12	10	5	5
M 4	3.614×10^{-6}	994	331	311	20	23	5	5
M 5	3.615×10^{-6}	994	331	311	20	23	5	5

Example 3: Consider the almost periodic linear problem studied by Stiefel and Bettis[5].
$\ddot{Z} + Z = 0.001e^{it}$, $Z(0) = 1$, $\acute{Z}(0) = 0.9995i$, with analytic solution.
Z(t) = cost+0.0005tsint+i(sin t-0.0005t cost).

This solution represents motion on a perturbed circular orbit with the distance from the origin |Z(t)|. We have computed solution to this problem using our fourth order method. We use the system

$$\ddot{u} + u = 0.001\cos t,\ u(0) = 1,\ \acute{u}(0) = 0,\ \ddot{v} + v = 0.001\sin t,\ v(0) = 1,\ \dot{v}(0) = 0.9995,$$

Where, Z = u+iv. After computing u and v, we have calculated |Z| at t = 40π with stepsize h = π/4. Result are presented in Table 7-9 and comparison of approximation in Table 10.

Table 7: Comparison of the method for example 3 with with Tol = 10^{-2}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	5.050×10^{-2}	967	322	321	1	0	1	1
M 2	1.311×10^{-2}	967	322	321	1	0	1	1
M 3	5.155×10^{-4}	967	322	321	1	0	1	1
M 4	1.180×10^{-2}	967	322	321	1	0	1	1
M 5	6.879×10^{-3}	967	322	321	1	0	1	1

Table 8: Comparison of the method for example 3 with with Tol = 10^{-4}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	1.234×10^{-2}	3853	1284	1281	3	0	3	3
M 2	3.087×10^{-3}	3853	1284	1281	3	0	3	3
M 3	2.424×10^{-6}	3853	1284	1281	3	0	3	3
M 4	3.088×10^{-3}	3853	1284	1281	3	0	3	3
M 5	1.545×10^{-3}	3853	1284	1281	3	0	3	3

Table 9: Comparison of the method for example 3 with with Tol = 10^{-6}

Methods	Error	FCN	NOST	NSST	NFST	NCST	NDIV	NHST
M 1	6.161×10^{-3}	7693	2564	2560	4	0	4	4
M 2	1.540×10^{-3}	7693	2564	2560	4	0	4	4
M 3	8.542×10^{-7}	7693	2564	2560	4	0	4	4
M 4	1.543×10^{-4}	7693	2564	2560	4	0	4	4
M 5	7.694×10^{-4}	7693	2564	2560	4	0	4	4

Table 10: Comparison of the approximation produced at Z=40π with h=π/4 initially.Exact value is Z(40π)=1.0019720

Methods	TOL=10×10^{-2}	TOL=10×10^{-4}	TOL=10×10^{-6}
M 1	1.0524749	1.0143116	1.0081329
M 2	1.0150825	1.0050591	1.0035116
M 3	1.0024875	1.0019744	1.0019711
M 4	0.9901767	0.9988838	1.0004287
M 5	1.0088506	1.0035170	1.0027414

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