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Fourier Transform Solution of the Semi-linear Parabolic Equation

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Abstract: The semi-linear heat equation models many physical processes, which include time-dependent, irreversible processes such as that of conduction, chemical reactions and biological flow problems. We have considered the one dimensional semi-linear heat equation and a piecewise continuous integrable function f(x, t). Using a Fourier cosine transform and a backward inverse operation we determined the solution to the problem.

Key words: Semi-linear equation, fourier transform, backward inverse operation

INTRODUCTION

We consider the problem

$$\frac{\partial u}{\partial t} - \Delta u = f(x,t,u) \qquad t > 0,\ x \in R^n$$

This equation models many physical processes, which include time-dependent, irreversible processes such as that of conduction, chemical reactions and biological flow problems. Thus many authors in various forms have investigated the problem imposing different boundary conditions which suit the situations at hand and using various forms of f (x, t, u). The type of solution one gets depends on the form of f(x, t, u) and the boundary conditions. The solution u(x,t) can be defined for all positive t, in which case we call it a global solution or unbounded in finite time, in which case we say it blows up.

Escobedo and Herrero[1] and Fujita and Watanabe[2] considered

$$\frac{\partial u}{\partial t} - \Delta u = \lambda u^q \quad (0 < q < 1),\ \lambda > 0 \quad (t,x) \in (0,T)\, x\, \Omega \qquad 1(a)$$

$$u = 0 \qquad (t, x) \in (0,T) \times \delta\Omega \qquad (b)$$

$$u(0,x) = u_0(x) \qquad (c)$$

where, Ω is a bounded open subset of R^N and established the existence and uniqueness of positive solution. Assuming that the initial value u_0 is non-negative, they found the solution as:

$$u(t) = T(t)u_0 + \int_\partial^t T(t\text{-}s)\, g\,(u(s))\, ds \qquad (2)$$

for all $t \in [0, T]$

Rossi[3] obtained the blow up rate for positive solution of:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \lambda u^p \qquad (0, 1)\ x\ (0, T) \qquad 3(a)$$

with boundary conditions

$$\frac{\partial u(1,t)}{\partial x} = u^q(1,t) \qquad (b)$$

$$\frac{\partial u(0,t)}{\partial x} = 0. \qquad (c)$$

In this case one has a non-linear term at the boundary and a reaction term in the equation. If $\lambda>0$, these 2 terms compete and blow up phenomenon occurs if p<2q-1 or p = 2q-1.

They established that if p<2q-1 or p = 2q-1 the blow up rate is given by:

$$u(1, t)\,\mu\,(T\text{-}t)^{-1/_{p\text{-}1}} \qquad (4)$$

When $\lambda>0$, Chipot *et al.*[4] and Lopez *et al.*[5] proved the existence and regularity of solution for initial data that satisfies compatibility condition. They found out that the solution of eqn. (3) only exists for a finite period of time. Cazenave *et al.*[6] introduced the concave and convex term in the equation in the form:

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$$u_t - \Delta u = \lambda u^q + u^p \quad (t,x) \in (0,T) \times \Omega \qquad 5(a)$$

$$u = 0 \qquad (t,x) \in (0,T) \times \delta\Omega \qquad (b)$$

$$u(0,x) = u_0(x) \qquad (c)$$

including a Dirichlet boundary condition. The non-linearity on the RHS is the sum of the concave and convex term with the non-linearity being singular at 0 (it is not Lipschitz because q<1). They showed that there exists a global solution if and only if there exists a weak solution of the stationary equation. Then Haraux[7] considered a linear parabolic equation with Lipschitz continuous boundary condition

$$u_t - \Delta u + a(t,x)u = 0 \quad \text{in } R^+ \times \Omega \qquad 5(a)$$

$$u = 0 \quad \text{in } R^+ \times \delta\Omega \qquad (b)$$

$$u(0,x) = u_0(x) \qquad (c)$$

and showed that there is a unique global solution. Using this he established that for

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u^3 - \lambda u = 0 \quad \text{in } R^+ \times (0,L) \qquad 7(a)$$

$$u(t,0) = u(t,L) = 0 \text{ on } R^+ \qquad (b)$$

then

$$\int u(0,x)\,u(t,x)\,dx < 0 \quad \text{for some } t > 0 \qquad (8)$$

In a recent work, Messaoudi[8] proved the local existence for the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} = |u|^{p-2}u \quad x \in I = (0,1), \quad 0 < t \qquad 9(a)$$

$$u(1,t) = 0 \qquad t \geq 0 \qquad (b)$$

$$u(x,0) = \phi(x) \qquad x \in I \qquad (c)$$

and showed that the solution blows up in finite time.

In the realm of the classical theory of differential equations, it can be shown imposing a Lipschitz condition on the non-linearity term f that the equation has a unique solution if $u_0(x) \in C^\infty(\Omega)$. On the other hand if $u_0(x)$ is a distribution with compact support, the system has no solution in the classical sense[9]. Thus Ifidon and Oghre[10] formulated a generalized function G(Q), using classical estimates and induction hypothesis over the order of the differential operators which defines the element of G(Q) to prove the existence, uniqueness as well as consistence results for the solution to the problem.

This study considers:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x,t)$$
$$u(x,0) = 0$$
$$\frac{\partial u(0,t)}{\partial t} = h(t)$$

and use an inverse process of Fourier transform to determine the solution u(x, t).

METHOD OF SOLUTION

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 10(a)$$

$$u(x,0) = 0 \qquad (b)$$

$$\frac{\partial u(0,t)}{\partial t} = h(t) \qquad (c)$$

Let u(x, t) be the solution and assume that both u(x, t) and f(x, t) are piecewise smooth and integrable over (0, ∞), then by Fourier cosine transform

$$u_c(s,t) = \sqrt{\frac{2}{\pi}}\int u(x,t)\cos sx\,dx \qquad (11)$$

Differentiating (11) with respect to t and substituting in (10)

$$\frac{\partial u_c}{\partial t}(s,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u_t(x,t)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(k\frac{\partial^2 u}{\partial x^2} + f(x,t)\right)\cos sx\,dx$$

$$= k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x,t)\cos sx\,dx$$

$$\frac{\partial u_c}{\partial t}(s,t) = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\,dx + F_c(s,t) \qquad (12)$$

where, F_c is the Fourier cosine transform of f(x, t).

Evaluating the integral on the right using integration by parts

$$\frac{\partial u_c}{\partial t}(s,t) = k\sqrt{\frac{2}{\pi}}\left[\left(\frac{\partial u}{\partial x}\cos sx + su\sin sx\right)\Bigg|_0^\infty - s^2\int_0^\infty u\cos sx dx\right] + F_c(s,t) \tag{13}$$

$$= -k\sqrt{\frac{2}{\pi}}h(t) - ks^2u_c(s,t) + F_c(s,t)$$

$$\frac{\partial u_c}{\partial t} + ks^2 u_c(s,t) = F_c(s,t) - k\sqrt{\frac{2}{\pi}}h(t) \tag{14}$$

This is a linear differential equation whose integrating factor is given as

$$R(x) = \exp(ks^2t)$$

$$\therefore u_c(s,t) = \int_0^t F_c(s,t)e^{-ks^2(t-\tau)}d\tau - k\sqrt{\frac{2}{\pi}}\int_0^t h(\tau)e^{-ks^2(t-\tau)}d\tau \tag{15}$$

Taking the inverse Fourier cosine transform

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\int_0^t F_c(s,t)e^{-ks^2(t-\tau)}d\tau - k\sqrt{\frac{2}{\pi}}\int_0^t h(\tau)e^{-ks^2(t-\tau)}d\tau\right]\cos sxds \tag{16}$$

Interchanging the order of integration, the first integral becomes

$$\sqrt{\frac{2}{\pi}}\int_0^t\left[\int_0^\infty F_c(s,t)e^{-ks^2(t-\tau)}d\tau\right]\cos sxds = \sqrt{\frac{2}{\pi}}\int_0^t\left[\int_0^\infty\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi,\tau)\cos s\xi d\xi\right]e^{-ks^2(t-\tau)}\cos sxdsd\tau \tag{17}$$

$$= \frac{2}{\pi}\int_0^t\int_0^\infty\int_0^\infty f(\xi,\tau)e^{-ks^2(t-\tau)}\cos s\xi\cos sxd\xi dsd\tau$$

$$= \frac{2}{\pi}\int_0^t\int_0^\infty\int_0^\infty f(\xi,\tau)e^{-ks^2(t-\tau)}\frac{1}{2}\left(\cos s(x-\xi)+\cos s(x+\xi)\right)d\xi dsd\tau$$

$$= \frac{1}{\pi}\int_0^t\int_0^\infty\int_0^\infty f(\xi,\tau)e^{-ks^2(t-\tau)}\left(\cos s(x-\xi)+\cos s(x+\xi)\right)d\xi dsd\tau$$

$$= \frac{1}{2\pi}\int_0^t\int_0^\infty\int_{-\infty}^\infty f(\xi,\tau)e^{-ks^2(t-\tau)}\left(\cos s(x-\xi)+\cos s(x+\xi)\right)d\xi dsd\tau \tag{18}$$

Using the fact that $\sin s(x-\xi)+\sin s(x+\xi)$ being odd then

$$\frac{1}{2\pi}\int_0^t\int_0^\infty\int_{-\infty}^\infty f(\xi,\tau)e^{-ks^2(t-\tau)}\left(\sin s(x-\xi)+\sin s(x+\xi)\right)d\xi dsd\tau = 0$$

Then equation (18) becomes

$$\int_0^t\int_0^\infty\left[\frac{1}{2\pi}\int_{-\infty}^\infty f(\xi,\tau)e^{-is(x+\xi)-ks^2(t-\tau)}ds + \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi,\tau)e^{-is(x-\xi)-ks^2(t-\tau)}ds\right]d\xi d\tau$$

$$= \int_0^t\int_0^\infty\left[G(x+\xi,t-\tau)+G(x-\xi,t-\tau)\right]f(\xi,\tau)d\xi d\tau \tag{19}$$

where,
$$G(x+\xi,t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(x+\tau)-ks^2(t-\tau)}ds \tag{20}$$

Also the second integral in (16) becomes

$$-k\sqrt{\frac{2}{\pi}}\int_0^t \frac{e^{-\frac{x^2}{4k(t-\tau)}}}{\sqrt{t-\tau}}h(\tau)d\tau \tag{21}$$

Hence by (19) and (21)

$$u(x,t)=\int_0^t\int_0^\infty\left[G(x+\xi,t-\tau)+G(x-\xi,t-\tau)\right]f(\xi,\tau)d\xi d\tau-k\sqrt{\frac{2}{\pi}}\int_0^t\frac{e^{-\frac{x^2}{4k(t-\tau)}}}{\sqrt{t-\tau}}h(\tau)d\tau$$

CONCLUSION

We have considered the one dimensional semi-linear heat equation and we have considered a piecewise continuous integrable function f(x, t). Using a Fourier cosine transform and a backward inverse operation we have determined the solution to the problem.

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