Journal of
Applied Sciences

Non-linear Solution of Function Generation of Planar Four-link Mechanisms by Homotopy Continuation Method

Tzong-Mou Wu

Department of Mechanical Engineering, Far East College, Tainan 744, Taiwan

Abstract: The automation design program of planar four-link mechanisms, including four-bar and slider-crank mechanisms, were presented. This program was developed on popular windows platform by Microsoft visual basic language. The program can manage maximum analytical precision points' synthesis problems of planar four-link mechanisms. We just input necessary desired specs, the link lengths and mechanism profile will quickly be determined by this program. Moreover, the animation simulation can also be performed in this program. Some examples were given in this study for illustration purpose.

Key words: Four-bar linkage, slider-crank mechanism, design, synthesis, CAD, programming, simulation

INTRODUCTION

Tao[1] developed a serial of graphical synthesis methods in 1964's. However, as we known, the graphical methods are not exact solutions. Also, the methods have some disadvantages such as the limitation of precision points and the inaccuracy of drawing and so on. The loop-closure equation had carried out by Freudenstein[2]. The Freudenstein's equation provided the ability to get exact solutions. Nevertheless, we still must use calculator, even computer, to help us to accomplish higher analytical precision points by Freudenstein's equation. Freudenstein's equation provided maximum 3 linear exact solutions and maximum 5 non-linear analytical solutions. There are different works that has been devoted to the courses of mechanism design until today[3-6].

In the process of solving mechanism design problems, some troublesome simultaneous equations would be generated, especially the simultaneous non-linear equations. Up to the present, we have already many different methods can deal with the simultaneous non-linear equations[7]. The initial guesses in these numerical methods are a big trouble. Good initial guesses can faster converge to the answers. Reversely, bad initial guesses perhaps imply divergence. It's a well-known common sense.

Homotopy continuation method was known as early as in the 1930's. This method was used by kinematician in the 1960's in the U.S. for solving mechanism synthesis problems. The latest development was done by Morgan[8] at GM. We also have three literatures by Morgan[9,10], Garcia[11] and Allgower[12]. Morgan's issues are better readable than others. Continuation method gives a set of certain answers and some simple iteration process to obtain our solutions more exactly. Wu[13,14] presented some techniques for combining Newton's, ancient Chinese algorithm and homotopy methods to avoid divergence on solving non-linear equation. Moreover, Wu[15] explored all the linear and non-linear solutions of the kinematics design problems as well as Wu[16] applied the homotopy continuation method to search all the roots of inverse kinematics problem of robot. This study attempt to use Freudenstein's equation and homotopy continuation method to solve the maximum analytical precision points' synthesis problems of planar four-link mechanisms. The program is developed on current popular windows platform by Microsoft visual basic language. Further, the program is provided in this study. (http://home.kimo.com.tw/tm.wu/Programs/Programs.htm):

Freudenstein's equation and homotopy continuation method: Consider Fig. 1, the planar four-bar linkage mechanism has the following Freudenstein's equation

$$K_1 \cos \phi_i + K_2 \cos \psi_i + K_3 = -\cos(\phi_i - \psi_i) \qquad (1)$$

Where:

$$K_1 = \frac{Z_1}{Z_4}, K_2 = -\frac{Z_1}{Z_2}, K_3 = \frac{Z_3^2 - Z_1^2 - Z_2^2 - Z_4^2}{2Z_2Z_4}, i \in N \quad (2)$$

Observing Eq. 1 and 2, the givens are (ϕ_i, Ψ_i) and unknowns are (Z_1, Z_2, Z_3, Z_4). Since Z_1 is a scale, it can be chosen arbitrarily. The real unknowns therefore become (Z_2, Z_3, Z_4). Hence we only have three equations to solve maximum three unknowns linearly by Cramer's rule or Gauss elimination method, etc.
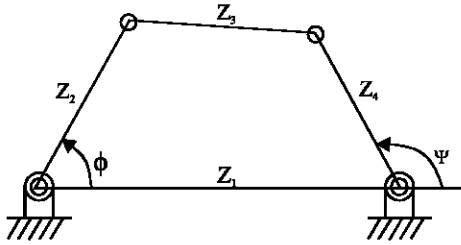
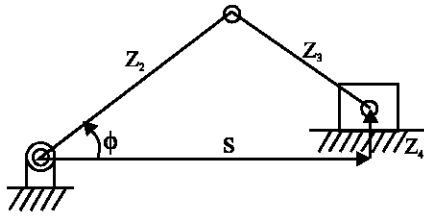Fig. 1: The planar four-bar linkage mechanism

Fig. 2: The planar slider-crank mechanism

However, according to the Chebyshev spacing in the actual synthesis problems, we have two free variables can be assigned, i.e. (ϕ_0, Ψ_0), which are the initial position of our desired input and output design intervals. In other words, we total have five unknowns $(Z_2, Z_3, Z_4, \phi_0, \Psi_0)$ to be determined. It will yield Eq. 1 to be

$$K_1 \cos(\phi_0 + \phi_{0i}) + K_2 \cos(\psi_0 + \psi_{0i}) + K_3$$
$$= -\cos(\phi_0 + \phi_{0i} - \psi_0 - \psi_{0i}) \qquad (3)$$

Theoretically, Eq. 3 can satisfy infinite positions (ϕ_i, Ψ_i) just by the five variables $(Z_2, Z_3, Z_4, \phi_0, \Psi_0)$. We can use so-called overlay method to synthesize this kind of mechanism. In fact, the successful probability is very lower. With the natural constraints of planar four-link mechanisms, we only can get 2 or 3 precision points for linear solutions and 4 or 5 precision points for nonlinear solutions by Freudenstein's equation. To ensure the existence of solutions, the n unknowns must correspond to n equations. This is Newton's fundamental theorem of algebra. So we can solve Eq. 3 analytically up to maximum 5 input-output position relationships, i.e. 5 precision points. But, the five unknown variables in Eq. 3 are coupled each other. We have to apply numerical technique to solve them. The Newton-Raphson is a famous method. The defect is acquirement of good initial guesses. With the aid of homotopy continuation approach, we will more correct and quicker to obtain the results.

Firstly, we rewrite Eq. 3 as function f$(Z_2, Z_3, Z_4, \phi_0, \Psi_0)$, then we assume a new controllable/given function g$(Z_2, Z_3, Z_4, \phi_0, \Psi_0)$. The homotopy continuation function therefore is defined as:

$$H(i) \equiv tf(Z_2, Z_3, Z_4, \phi_0, \psi_0) +$$
$$(1-t)g(Z_2, Z_3, Z_4, \phi_0, \psi_0) = 0 \qquad (4)$$

Where:

$$f = K_1 \cos(\phi_0 + \phi_{0i}) + K_2 \cos(\psi_0 + \psi_{0i})$$
$$+K_3 + \cos(\phi_0 + \phi_{0i} - \psi_0 - \psi_{0i})$$
$$g \equiv K_1 \cos(\psi_0) + K_2 \cos(\psi_0) + K_3 +$$
$$\cos(\phi_0 - \psi_0) - 2\cos(\phi_{00}) - 2 \qquad (5)$$

The last two terms $-2\cos(\phi_{00})-2$ of function g mean the correction dependent on the controllable initial guesses $(K_1, K_2, K_3, \Psi_0) = (1, 1, 1, 0)$ for 4 precision points. As well as $(K_1, K_2, K_3, \phi_0, \Psi_0) = (1, 1, 1, \phi_{00}, 0) = (1, 1, 1, 0, 0)$ for 5 precision points.

What we have to do is change the homotopy parameter t from 0 to 1 to get the answer of f = 0. By the test, the iteration times do not need very large. We just only need to iterate 100 times to yield the converge results. Of course, the iteration procedure of Newton-Raphson method is still necessary. In many situations, the convergence speed of Newton-Raphson combined with homotopy is faster than only using the Newton-Raphson method.

Secondly, from Fig. 2, the Freudenstein's equation of planar slider-crank mechanism is:

$$K_1 S_i \cos\phi_i + K_2 \sin\phi_i - K_3 = S_i^2 \qquad (6)$$

Where:

$$K_1 = 2Z_2, K_2 = 2Z_2 Z_4, K_3 = Z_2^2 - Z_3^2 + Z_4^2, i \in N \qquad (7)$$

Similarly, the homotopy continuation function for the planar slider-crank mechanism is defined as:

$$H(i) \equiv tf(Z_2, Z_3, Z_4, \phi_0, S_0) +$$
$$(1-t)g(Z_2, Z_3, Z_4, \phi_0, S_0) = 0 \qquad (8)$$

Where:

$$f = K_1(S_0 + S_{0i})\cos(\phi_0 + \phi_{0i}) +$$
$$K_2 \sin(\phi_0 + \phi_{0i}) - K_3 - (S_0 + S_{0i})^2$$
$$g \equiv K_1 S_0 \cos(\phi_0) + K_2 \sin(\phi_0)$$
$$-K_3 - S_0^2 - \sin(\phi_{00}) - 1 \qquad (9)$$

EXAMPLES

Example 1: Synthesis of planar four-bar mechanism with 2 precision points: If we would like to synthesize 2 precision points from Eq. 1, one free choice should be given because of we have 2 equations and 3 unknowns (Z_2, Z_3, Z_4) or (K_1, K_2, K_3). One variable of Z_2, Z_3 or Z_4 will

be assigned to determine following simultaneous linear equations

$$\begin{cases} K_1 \cos\phi_1 + K_2 \cos\psi_1 + K_3 = -\cos(\phi_1 - \psi_1) \\ K_1 \cos\phi_2 + K_2 \cos\psi_2 + K_3 = -\cos(\phi_2 - \psi_2) \end{cases} \quad (10)$$

We design the free choice as Z_2 in this study. Of course, Z_1 also is the free choice in all cases of planar four-bar linkage mechanisms. We give the input-output data, (ϕ_1, Ψ_1) and (ϕ_2, Ψ_2), in this example as enter the values and free choices $Z_1 = 5.2$, $Z_2 = 0.8$ shown in Fig. 3. The results are the same with Tao[1] contribution.

Example 2: Synthesis of planar slider-crank mechanism with 2 precision points: In general, planar slider-crank mechanism with 2 precision points means crank and piston system. Similarly, if we desire to synthesize this kind of mechanism. Consider Eq. 6, we can choose the free variable from Z_2 (crank length) or Z_4 (offset from fixed pivot to slider) to determine simultaneous linear Eq. 11. This study provides these two choices. In this example, we let offset $Z_4 = -2$ to compare with Tao[1] literature.

$$\begin{cases} K_1 S_1 \cos\phi_1 + K_2 \sin\phi_1 - K_3 = S_1^2 \\ K_1 S_2 \cos\phi_2 + K_2 \sin\phi_2 - K_3 = S_2^2 \end{cases} \quad (11)$$

The answers are also the same with Tao[1]. Note that, in Fig. 4, Z_1 denotes S_i displacement in the planar slider-crank mechanism, drawn in Fig. 2. We already give them values in "Si or S" column of Fig. 4. So we do not need to assign its data again. On the other hand, the scale in this mechanism should be the real dimension of the slider.

Example 3: Synthesis of planar four-bar mechanism with 3 precision points: Three precision points are the limitation of the planar four-link mechanism syntheses, including four-bar and slider-crank mechanisms, with simultaneous linear equations. We do not need to assign redundant free variable except for the fixed link Z_1 for four-bar linkage. The simultaneous linear equations in this example are

$$\begin{cases} K_1 \cos\phi_1 + K_2 \cos\psi_1 + K_3 = -\cos(\phi_1 - \psi_1) \\ K_1 \cos\phi_2 + K_2 \cos\psi_2 + K_3 = -\cos(\phi_2 - \psi_2) \\ K_1 \cos\phi_3 + K_2 \cos\psi_3 + K_3 = -\cos(\phi_3 - \psi_3) \end{cases} \quad (12)$$

Now, we run this example as the type of enter the function, shown in Fig. 5. We supply the design information as:

$$y = \sin x, \ 0° \le x \le 90°, \ \phi_0 = 97°, \ \Delta\phi = 120°,$$
$$\psi_0 = 60°, \ \Delta\psi = 60°, \ Z_1 = 52.5 \quad (13)$$
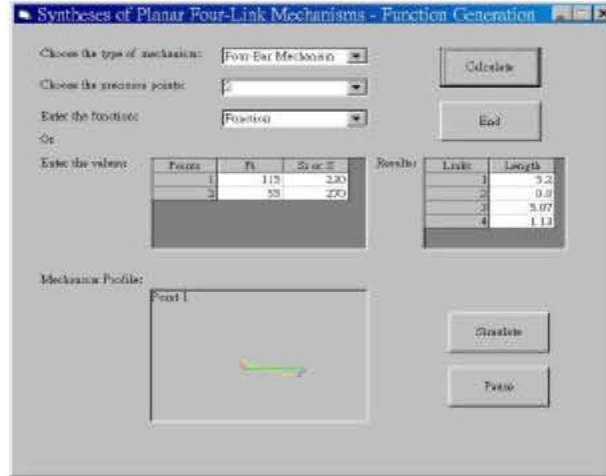

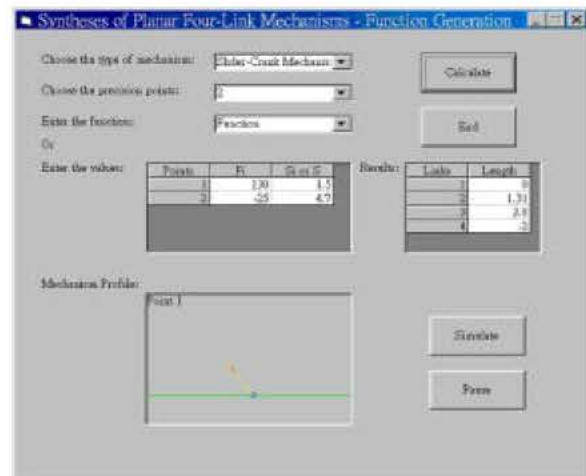Fig. 3: Synthesis of planar four-bar linkage mechanism with 2 precision points

Fig. 4: Synthesis of planar slider-crank mechanism with 2 precision points
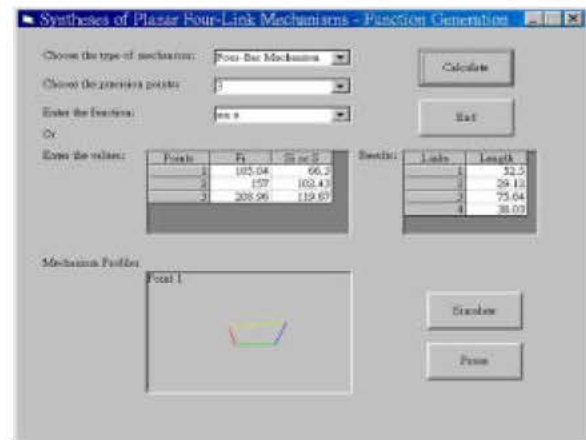

Fig. 5: Synthesis of planar four-bar linkage mechanism with 3 precision points
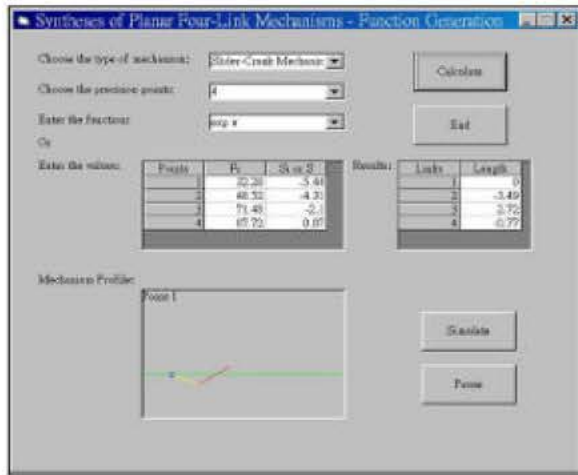
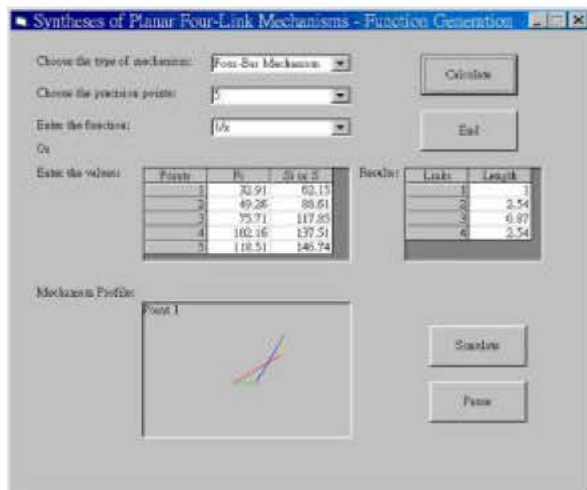Fig. 6: Synthesis of planar slider-crank mechanism with 4 precision points

Fig. 7: Synthesis of planar four-bar linkage mechanism with 5 precision points

Where, ϕ_0, Ψ_0 and Z_1 are arbitrary. Specifically, ϕ_0 and Ψ_0 are arbitrary for linear solutions of Eq. 12 by means of using Chebyshev spacing method to yield (ϕ_1, Ψ_1), (ϕ_2, Ψ_2) and (ϕ_3, Ψ_3). We can also change any value (ϕ_i, Ψ_i) by "Click" and input data in the "Values Grid" in Fig. 5 to meet special location requirement such as positioning machine.

The computation results of links length, shown in Fig. 5, are checked by Sandor[5]. They are the same.

Example 4: Synthesis of planar slider-crank mechanism with 4 precision points: Exceed 4 precision points, Eq. 1 and 6 will become non-linear. This is discussed in previous section. We especially apply the homotopy continuation method to handle this problem. Also, we

define two homotopy functions (4) and (8), respectively in this paper to solve the simultaneous non-linear equations.

We give the following design specs to run this example. The answers are shown in Fig. 6. The automation program takes us just 0.47 sec in merely AMD-K6/2-500 CPU.

$$y = e^x, \ 0 \le x \le 1, \ \phi_0 = 30°, \Delta\phi = 60°, \ \Delta S = 6$$

$$(14)$$

Where, we still have one free parameter ϕ_0 to be chosen arbitrarily in all problems of 4 precision points syntheses.

Example 5: Synthesis of planar four-bar mechanism with 5 precision points: By the similar manner, we provide a set of design data as following to synthesize the planar four-bar linkage mechanism with 5 precision points

$$y = \frac{1}{x}, \ 1 \le x \le 2, \ \Delta\phi = 90°, \ \Delta\psi = 90°, \ Z_1 = 1 \qquad (15)$$

We can also obtain these design data from Hartenberg's book[4]. Owing to the computation of this program, we find the inaugural Chebyshev's data x_1 to x_5 are puzzle in Hartenberg's book. They seem wrong.

The final results of this example are shown in Fig. 7. The program takes us just only about 0.64 sec to run this example. It presents that the homotopy continuation method has high efficiency in solving simultaneous non-linear equations.

CONCLUSIONS

All mechanism design problems should solve simultaneous equations except for graphical methods. However, the graphical methods can not obtain the better accurate results. We must use analytical methods to determine the exact solutions. Whatever we solve what kinds of simultaneous equations and no matter how many precision points are assigned, we always have two types of simultaneous equations. They are linear and non-linear. This paper applies Gauss elimination and Homotopy continuation methods to solve simultaneous linear and non-linear equations, respectively.

In analytical solving the planar four-bar and slider-crank mechanisms, we have 2 to 5 precision points can be assigned. The giving of 2 or 3 precision points can generate simultaneous linear equations. However, 4 or 5 precision points will generate simultaneous non-linear equations. The solutions for these two types are not difficult for this program. Over 5 precision points, we

should use graphical method such as overlay method. It is not a closed form method. It is a trial and error technique and not guarantees the solutions. As we known, more free variables mean more answers for our problems. Reversely, we can equate less free choices to lesser results for the design problems. So it is not easy for us to certainly say what kind of design problems is nice. May be, we should decide them by the real needs.

This study develops a PC's program on windows platform by Microsoft visual basic language to solve the design problems of planar four-bar and slider-crank mechanisms. Some programming techniques and debug measures are applied and included in the program. Furthermore, the animation simulation artifice is also been combined in this program. From this program, we can quickly and exactly obtain our desired requirements and view the animation simulation real-time. It is hoped that the study presented here will contribute towards progress in the kinematics syntheses of mechanisms and provide some programming approaches for the scientists or engineers.

REFERENCES

1. Tao, D.C., 1964. Applied Linkage Synthesis. Addison-Wesley Publishing Company, Inc., London, Printed in USA.
2. Freudenstein, F., 1954. An analytical approach to the design of four-link mechanisms. Transactions of ASME, 76: 483-492.
3. Denavit, J. and R.S. Hartenberg, 1955. Approximate synthesis of four-bar linkages. Transactions of ASME., 77: 853-861.
4. Hartenberg, R.S. and J. Denavit, 1964. Kinematic Synthesis of Linkages. McGraw-Hill Series in Mechanical Engineering, McGraw-Hill, New York.
5. Sandor, G.N. and A.G. Erdman, 1997. Advanced Mechanism Design. Prentice-Hall, New Jersey.
6. Erdman, A.G., 1993. Modern Kinematics-Developments in the Last Forty Years. John Wiley and Sons, Inc., New York.
7. Epperson, J.F., 2002. An Introduction to Numerical Methods and Analysis. John Wiley and Sons, Inc., New York.
8. Morgan, A.P., 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. GM Research Publication, GMR 3651.
9. Morgan, A.P., 1987. Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems. Prentice-Hall Book Company, Inc., Englewood Cliffs, New Jersey.
10. Morgan, A.P. and C.W. Wampler, 1990. Solving a planar four-bar design problem using continuation. ASME J. Mechanical Design, 112; 544-550.
11. Garcia, C.B. and W.I. Zangwill, 1981. Pathways to Solutions, Fixed Points and Equilibria. Prentice-Hall Book Company, Inc., Englewood Cliffs, New Jersey.
12. Allgower, E.L. and K. Georg, 1990. Numerical Continuation Methods, An Introduction. Springer-Verlag, New York.
13. Wu, T.M., 2005. A study of convergence on the Newton-homotopy continuation method. Applied Mathematics and Computation, (In Press).
14. Wu, T.M., 2005. A modified formula of ancient Chinese algorithm by the homotopy continuation technique. Applied Mathematics and Computation, (In Press).
15. Wu, T.M., 2004. The kinematics design problems. J. Applied Sci., 4: 398-405.
16. Wu, T.M., 2005. Searching all the roots of inverse kinematics problem of robot by homotopy continuation method. J. Applied Sci., 5: 666-673.