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On Directed Metric Spaces

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Abstract: The concept Metric Spaces, as an abstraction of the distance function has as part of the axioms the symmetric property d(x,y)=-d(y,x). We consider an entirely different case of d(x,y)=-d(y,x), the resulting space is baptized a directed-metric space. We observe that this is an abstraction of the displacement function. We examine five examples of such space closely before stating and proving general theorems about them. For example we prove that both the subspaces Z and Q of integers and rationals, respectively are dense in R. We conjecture that every infinite subset of R is dense.

Key words: Metric spaces, distance function

INTRODUCTION

We proceed by defining first a metric space

Definition 1: Let X be a set, a function d defined on X×X into the positive real numbers is a metric on X. If the following are satisfied.

(a) $d(x,y)\geq 0\ \forall x, y\in X$ (non-negativity)
(b) $d(x,y)=0$ iff $x=y$ (identity)
(c) $d(x,y)=d(y,x)$ (symmetry)
(d) $d(x,y)\leq d(x,z)+d(z,y), \forall x,y,z\in X$ (triangular inequality)

(X,d) is said to be a metric space. Semi-metric and Pseudo-metric spaces have emerged from this and extensive work have been done on them[1-5]. We consider here an entirely different space, an abstraction of the displacement function which we define thus.

Definition 2: Let X be a set, suppose we define dd on X×X such that $dd:X\times X\to R$ satisfies the followings:

(a) $dd(x,y)=-dd(y,x)$
(b) $|dd(x,y)|\leq|dd(x,z)|+|dd(z,y)|\forall x,y,z\in X$

If we call dd a directed-metric we say (X,dd) is a directed-metric space.

The fundamental notion here is the non-symmetry property i.e $dd(x,y)=-dd(y,x)$. We observe that the identity property is deducible from the non-symmetry for if x=y we have that $dd(x,y)=-dd(y,x)$ reduces to $dd(x,x)=0$ which is the identity property. The function dd as a directed distance function indicates that the sign depends on the direction.

Example 1: Consider R the set of all real number with $dd(x,y)=x^n-y^n\forall x,y \in Rn \in N$ then dd is a directed metric on R and (R, dd) is a directed metric space.
Proof:
$dd(x,y)=x^n-y^n\forall x,y \in R, n \in N$
(a) We have that $dd(x,y)=x^n-y^n=-(y^n-x^n)=-dd(y,x)$
thus $dd(x,y)=-dd(y,x)$
(b) $|dd(x,y)|=|x^n-y^n|=|x^n-z^n+z^n-y^n|$
$=|(x^n-z^n)+(z^n-y^n)|\leq|(x^n-z^n)|+|(z^n-y^n)|$
$=|dd(x,z)|+|dd(z,y)|$
i.e $|dd(x,y)|\leq|dd(x,z)|+|dd(z,y)|$

Example 2: Let X be any set, if we define

$$dd(x,y)=\begin{cases}-1 & x>y\\ 0 & x=y\\ 1 & x<y\end{cases}$$

$\forall x,y,\in X$, (X,dd) make a discrete directed metric space. This example shows that an ordered set can be made into a directed-metric space.
Proof:
(a) Suppose x=y, dd(x,y)=0 and there is nothing to prove. Now if x>y, dd(x,y)=-1 thus we have that dd(y,x)=1 (since y<x and dd(x,y)=-dd(y,x)). Also if x<y, dd(x,y)=1, thus dd(y,x)=-1 (i.e. y>x) which implies dd(x,y)=-dd(y,x)
(b) Suppose x=y, dd(x,y)=0
dd(x,z)=0, dd(z,y)=0, thus
$|dd(x,y)|\leq|dd(x,z)|+|dd(z,y)|$
if x<y, dd(x,y)=1
dd(x,z)=1 and dd(z,y)=1
$|dd(x,y)|\leq|dd(x,z)|+|dd(z,y)|$
if x>y,

dd(x,y)=-1, dd(x,z)=-1 and dd(z,y)=-1
We have that
$|dd(x,z)| \leq |dd(x,z)|+|dd(z,y)|$

thus dd so define is a directed-metric.

Example 3: Consider the set of points of a complex plane C. We define the directed-metric
$dd(z_1,z_2)=(|z_1|^n-|z_2|^n)\ \forall z_1,z_2\in C,\ n\in Z$
(C,dd) is a directed metric space.
Proof:
(a) $dd(z_1,z_2)=(|z_1|^n-|z_2|^n)=-(|z_2|^n-|z_1|^n)=-dd(z_2,z_1)$
(b) $dd(z_1,z_2)=(|z_1|^n-|z_2|^n)$
$dd(z_1,z)=(|z_1|^n-|z|^n)$
$dd(z,z_2)=(|z|^n-|z_2|^n)$
Thus $|dd(z_1,z_2)|=(|z_1|^n-|z|^n + |z|^n-|z_2|^n)\leq|(|z_1|^n-|z|^n)|+|(|z|^n-|z_2|^n)|$
$|dd(z_1,z_2)|\leq|dd(z_1,z)|+|dd(z,z_2)|$

dd so defined is a directed metric.

Example 4: $dd(x,y)=\int_a^b (x(t)-y(t))dt$ defines a directed-metric space on the space C[a,b].
Proof:
(a) Now $dd(x,y)=\int_a^b (x(t)-y(t))dt=-\int_a^b (y(t)-x(t))dt=-dd(y,x)$
(b) If suffices to prove that $|dd(x,z)|\leq|dd(x,y)|+|dd(y,z)|$
$|dd(x,z)|=|\int_a^b (x(t)-z(t))dt|=|\int_a^b (x(t)-y(t)+y(t)-z(t))dt|$
$= |\int_a^b (x(t)-y(t)+y(t)-z(t))dt|\leq|\int_a^b (x(t)-y(t))dt|+|\int_a^b (y(t)-z(t))dt|=|dd(x,y)|+|dd(y,z)|$
Thus dd is a directed metric.

ORDERING OF POINTS IN THE PLANE

Definition 3: Let x,y be points in the n-dimensional Euclidean plane i.e. $x=(x_1,x_2, \dots , x_n)$;
$y=(y_1,y_2, \dots ,y_n)$
x=y, iff $x_j=y_j\ \forall j$
x<y iff $x_1<y_1$
or $x_1=y_1$ but $x_2<y_2$
or $x_1=y_1$ and $x_2=y_2$ but $x_3<y_3$
or $x_1=y_1$, $x_2=y_2$ and $x_3=y_3$ but $x_4<y_4$ etc.
i.e x<y iff $x_j<y_j$ where, j is the smallest positive integer such that $x_j \not> y_j$ otherwise x>y.

Example 5: The set of all ordered n-tuples of real numbers (i.e. n-dimensional Euclidean space R^n) and the metric

$$dd(x,y)=\begin{cases}+\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} & \text{if } x<y\\ 0 & \text{if } x=y\\ -\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} & \text{if } x>y\end{cases}$$

defines a directed metric on (R^n,dd)
Proof:
(a) To prove that dd(x,y)=-dd(y,x)
Suppose x<y

$$dd(x,y)=+\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

also

$$dd(y,x)=-\sqrt{\sum_{i=1}^{n}(y_i-x_i)^2}$$

since $y>x) = -\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = -dd(x,y)$

If x=y, dd(x,y)=0 and there is nothing to prove

Now if x>y, $dd(x,y)=-\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$

$$dd(y,x)=+\sqrt{\sum_{i=1}^{n}(y_i-x_i)^2}$$

since $y<x) = +\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = -dd(x,y)$

(b) The problem reduces to proving that $|dd(x,z)|\leq|dd(x,y)|+|dd(y,z)|$
if x<z,

$$|dd(x,z)|=\left|+\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}\right|$$

$$=\left|+\sqrt{\sum_{i=1}^{n}(x_i-y_i+y_i-z_i)^2}\right|$$

$$=\left|+\sqrt{\sum_{i=1}^{n}[(x_i-y_i)+(y_i-z_i)]^2}\right|$$

$$=\left|+\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2}\right|$$

$$\leq\left|+\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right|+\left|+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right|$$

$$\leq|dd(x,y)|+|dd(y,z)|$$

Now if we suppose that x=z, $0\leq 0+0$ and the result follows.
Suppose x>z,

$$|dd(x,z)| = \left|-\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}\right|$$

$$= \left|-\sqrt{\sum_{i=1}^{n}(x_i-y_i+y_i-z_i)^2}\right|$$

$$= \left|-\sqrt{\sum_{i=1}^{n}[(x_i-y_i)+(y_i-z_i)]^2}\right|$$

$$= \left|-\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2}\right|$$

$$\leq \left|-\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right| + \left|-\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right|$$

$$\leq |dd(x,y)| + |dd(y,z)|$$

TOPOLOGICAL CONCEPT IN DIRECTED METRIC SPACE

Definition 4: Open, closed balls and boundary points.
Let (X,dd) be a directed-metric space and 'a' a fixed point, r a positive real number (r>a) we define an open ball.

(I) $B(a,r)=\{x \in X \mid dd(a,x)<r\}$ and
(ii) A closed ball $\underline{B}(a,r)=\{x \in X \mid dd(a,x) \leq r\}$ and the boundary points as
(iii) $Bp=\{x \in X \mid dd(a,x)=r\}$.

We see that relationship $Bp=\underline{B}(a,r)-B(a,r)$ is preserved.

Definition 5: A sunset Y of a directed-metric space (X,dd) is said to be dense in X if the closure of Y is the same as X.

Definition 6: A directed-metric space (X,dd) is said to be seperable if it contains a countable dense subsets.
Observe that, in R^1

$B(a,r)=(-\infty,a+r)$
while, $\underline{B}(a; r)=[-\infty,a+r]$
Thus $[-\infty,a+r]-(\infty,a+r)=a+r$
Also in R^2
$B((a,b),r)=\{(x,y) \mid dd((a,b),(x,y))<r\}$
$\underline{B}((a,b),r)=\{(x,y) \mid dd((a,b),(x,y)) \leq r\}$
while, $Bp=\{(x,y) \mid dd((a,b),(x,y))=r\}$

Remark 1: An open ball is an open set
Proof:
From the definition, we need to show that every point of the ball has a neighborhood which is contained in it i.e. Let B(a;r) be an open ball, we now have to show that $B(x,\in) \subset B(a;r)$
Let $y \in B(x,\in)$, then $dd(x,y)<\in$-r-dd(a,x)
and from triangle inequality. $dd(a,y) \leq dd(a,x)+dd(x,y)$ $<dd(a,x)+r-dd(a,x)<r$. This show that $y \in B(a;r)$ and hence $B(x,\in) \subset B(a,r)$ and the result follows.

Alternatively: We know that an open ball is defined as $B(a;r)=(-\infty,a+r)$ and hence for each point $q \in B(a;r)$ we may choose Sq=B(a;r) which is contained in the open ball.

Remark 2: A closed ball is neither close nor open
Proof:
$\underline{B}(a;r)$ is not open since a+r is not an interior point of $(-\infty,a+r)$. Also if we set a real number $q=a+r+\in$, $q \notin B(a;r)$ we know that $\underline{B}(a;r) \cap (G \setminus \{q\}) \neq \phi$ for any open ball G, thus q is an accumulation point of $\underline{B}(a;r)$. Hence $\underline{B}(a;r)$ is not close and the result follows.

Remark 3: Z is dense in R
Proof:
If suffices to show that $Z \cap (B(a;r)-\{q\}) \neq \phi \forall q \in R$
Suppose we assume on the contrary that $Z \cap (B(a;r))-\{q\})=\phi$
We have that $z \notin (B(a;r)-\{q\}), q \in R$
i.e. (B(a;r)-{q}) has no integers which is a contradiction.

Remark 4: Q is dense in R
Proof:
Since $Z \subset Q$ and $Z \cap (B(a;r)-\{q\}) \neq \phi \forall q \in R$
definitely $Q \cap (B(a; r)-\{q\}) \neq \phi \forall q \in R$ and Q is dense in R

Remark 5: R is separable in Z and Q
Proof:
The set R is separable since it contains countable dense subsets Z and Q.

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