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Killing Forms and Applications

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Abstract: Characterizations of Semi-simple algebra were initiated by Cartan. In recent years, semi-simple Lie algebras have been characterized with the help of Killing forms. In this study we have made an attempt to define generalized killing forms and have applied these to the question of existence of Lagrangians in a physical system.

Key words: Killing Forms, Semi-simple Algebra, Solvable Lie Groups, Lie Algebra, Lagrangians, Yang-Mills Equation, Euler-Lagrange Equations

INTRODUCTION

Killing Forms (KFs) play a very important role in characterizing semi-simple algebras[1]. One of these schemes is the Cartan's criterion that states that Lie algebra is semi-simple if and only if its KF is non-degenerate. Recall: a Lie algebra has an extra structure called the Lie bracket or Lie product which has close link with calculus of manifolds.

BASIC DEFINITIONS AND USEFUL RESULTS

Bilinear forms: Let V_n (F) be an n-dimensional linear space defined over the scalar field F. Then the mapping:

$$T: V_n \times V_n \to F$$

with the following axioms is called a bilinear form:

$$\text{(i) } T(\alpha u_1+\beta u_2, v) = \alpha T(u_1, v), +\beta T(u_2, v)$$
$$\text{(ii) } T(u, \gamma v_1+\delta v_2) = \gamma\, T(u, v_1) + \delta T(u, v_2)$$

for all $u_1, u_2, v_1, v_2 \in V_n(F)$ and $\alpha, \beta, \gamma, \delta \in F$

The rank of a bilinear form T on V_n (F) denoted by rank (T) is defined to be the rank of any matrix representation of T. We say that T is degenerate or non-degenerate according as whether rank (T) < dim (V) or rank (T) = dim (V).

A bilinear from T on a linear space $V_n(F)$ is said to be symmetric if:

$$T(u, v) = T(v, u) \text{ for all } u, v \in V.$$

Theorem 1: Let T: $IR^n \times IR^n \to IR$ be a bilinear form and let A be a matrix representation of T. Then A is symmetric if and only if T is symmetric in IR^n.

Theorem 2: Let T be a symmetric bilinear form on a linear space V_n (F). Then V_n has a basis $\{v_i\}^n_{i=1}$ in which T is represented by a diagonal matrix i, e. T $(v_i, v_j) = 0$ if $i \neq j$.

Quadratic forms: A mapping q: $V_n(F) \to F$ is called a quadratic form if q(v) = T(v, v) for some symmetric bilinear form T on V_n. If T is represented by a symmetric matrix $A = (a_{ij})$ then q: $V_n(F) \to F$ is represented in the form:

$$q(X) = T(X, X) = X^t AX$$

$$= \begin{pmatrix} x_1 & x_2 & \dots & \dots & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & .. & a_{3n} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & .. & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ .. \\ x_n \end{pmatrix}$$

$$= \sum_{i,j} a_{ij}\, x_i\, x_j$$

$$= a_{11}x_1^2 + a_{22}\, x_2^2 + \dots\dots\dots + a_{nn}\, x_n^2 + 2 \sum_{i<j} a_{ij}\, x_i\, x_j$$

Note that, if A is diagonal, then it has the diagonal representation

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$$q(X) = X^t A X = a_{11} x_1^2 + a_{22} x_2^2 + \dots\dots\dots + a_{nn} x_n^2,$$

That is, the quadratic polynomial representing 'q' will contain no “cross-product” terms. For example, consider the mapping q: $IR^2 \to IR$ over the scalar field IR, defined by:

$$q(x, y) = ax^2 + 2hxy + by^2, \text{ where } a, b, h \in IR$$

Then q (x, y) satisfies all the conditions of quadratic forms. The symmetric matrix A of this quadratic form is $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ and in case h = 0 i.e. A is diagonal then $q(x, y) = ax^2 + by^2$.

Killing forms[2-4]**:** Let G be a Lie algebra Suppose X,Y are arbitrary elements of G. Then the operator.

$$adX: G \to G$$

defined by adX(Y) = [X,Y] is a linear transformation.

Recall: If V is a vector space over the field F, Then a mapping T of V(F) into V(F) is called a linear transformation (or a linear operator) if T satisfies the following conditions:

(i) If $v_1, v_2 \in V(F)$, then $T(v_1 + v_2) = Tv_1 + Tv_2$

(ii)If $v_1 \in V(F)$, $s \in F$, then $T(sv_1) = sTv_1$

To see that adX: G→G defined by adX(Y) = [X,Y] is a linear transformation.
We calculate: $adX(Y+Z) = [X,Y+Z], \forall\, Y, Z \in G$

$$\begin{aligned} &= X(Y+Z) - (Y+Z)X \\ &= XY+XZ - YX - ZX \\ &= (XY - YX) + (XZ - ZX) \\ &= [X, Y] + [X, Z] \\ &= adX(Y) + adX(Z) \end{aligned}$$

Also, $adX(\alpha Y) = [X, \alpha Y]\ \forall\, \alpha \in F$

$$\begin{aligned} &= X\alpha Y - \alpha YX \\ &= \alpha(XY - YX) \\ &= \alpha[X,Y] \\ &= \alpha adX(Y) \end{aligned}$$

Hence the operator adX: G→G is a linear transformation.

Note that X→adX is a representation of the Lie algebra G with G itself considered as linear space of the representation. The representation adX, called the adjoint representation, always provides a matrix representation of the algebra. For example, the adjoint representation of the algebra of SO(3) is given by:

$(M_i)_{jk} = C^j_{ik} = \epsilon_{ikj} = -\epsilon_{ijk}$, where, ϵ_{ijk} is antisymmetric in i,k.

Thus the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } L_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with structure constants ϵ_{ikl} being antisymmetric in i and k given by:

$$[L_j, L_k] = \epsilon_{jkl} L_l$$

are also the matrices of the adjoint representation.

The Killing form of a Lie algebra G is the symmetric bilinear form:

$$K(X,Y) = Tr\,(adX\, adY).$$

If $\{E_i\}^n_{i=1}$ is a basis in G for then $g_{ij} = K(E_i, E_j) = C^r_{is} C^s_{jr}$ is called the metric tensor for G where, the C^k_{is} are the structure constants of G.

A GENERALIZATION OF KILLING FORMS

We consider a connected and compact Lie group with corresponding Lie algebra G_0 having $\dim(G_0) = l_0$ and any arbitrary Lie algebra G_1 with $\dim(G_1) = l_1$. Then the direct sum $G = G_0 \oplus G_1$ is a Z_2-graded Lie algebra. Suppose $B = \{X\}^{l_0+l_1}_{j=1}$ is a basis of G with $\dim(G) = l_0 + l_1$. Then form by the mapping

$$K^*: G \times G \to IR \quad (1)$$

Now we define the generalized Killing form by symmetric bilinear form:

$$K^*(X_i, X_j) = \sum(-1)^{d_1} C^l_{ir} C^r_{jl}$$

where:

$$d_l = \begin{cases} 0, \text{ if and only if } X_l \in G_0 \\ 1, \text{ if and only if } X_l \in G_1 \end{cases}$$

is called a degree of X_l. Further X_l is said to be even or odd, respectively if $d_l = 0$ or 1.

APPLICATION OF KFs

Invariant quadratic forms defined originally by the then algebraists have close link with KFs of differential geometry[5,6]. Cartan's criterion on semi-simplicity of algebra may be stated in an equivalent form: a Lie algebra G is semi-simple if and only if det $(g_{ij}) \neq 0$ where, g_{ij} is the metric tensor for G. Thus we see that semi-simple algebras always admit a non-degenerate invariant quadratic form on G. The general result stated above on KFs may be used to test a set of differential equations for the existence of a Lagrangian of a physical system.

Let G be a 2-dimensional non-abelian solvable Lie group with corresponding Lie algebra G spanned by the basis $\{X_1, X_2\}$,

Where:

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with commutation relation

$$[X_1, X_2] = 2X_2$$

We consider the Yang-Mills equations associated with G:

$$D_a F^{ab} = 0 \qquad (2)$$

with D_a, the covariant derivative associated with G-valued connection γ_a and curvature F_{ab} representing Yang-Mills potentials and fields, respectively. The algebra involved here has a degenerate KF since $\det(g_{ij}) = 0$. On the other hand, consider the SL(2,C) (C is set of complex numbers) algebra with a basis $\{X_1, X_2, X_3\}$ given by:

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Analogously, we define the Yang-Mills field equations associated with the above 3-dimensional algebra:

$$D_a F^{ab} = 0 \qquad (3)$$

The algebra involved here has a non-degenerate KF since $\det(g_{ij}) \neq 0$. Applying variational principle on principal fibre bundles, we see that the above Yang-Mills equations associated with the Lie group G under consideration are exactly the Euler-Lagrange differential equations[7]. Since the corresponding Lie algebra G admits a non-degenerate KF. But the equation (2) fails to be Euler-Lagrange equations since the corresponding Lie algebra G does not admit a non-degenerate KF.

CONCLUSION

Let us consider the GKF. It is possible to require that K^* acts on the sub-space G_0 only. Consequently K^* reduces to the usual Killing form for the Lie algebra G_0. Recall: G_0 is compact in the sense of the group K is negative definite and hence G_0 is semi-simple. It indicates that a graded Lie algebra G can be made to possess a GKF which guarantees that a Lagrangian must exist for the theory under consideration.

REFERENCES

1. Hermann, R., 1966. Lie Groups For Physicists. W.A. Benjamin, Inc. New York, USA., pp: 14-17.
2. Choquet-Bruhat, Y., C. Dewitt-Morette and M. Dillard-Bleick, 1977. In: Analysis, Manifolds and Physics. North-Holland Publishing Company. Netherlands, pp: 152-168.
3. Hamermesh, M., 1962. Group Theory and its Application in to Physical Problems. Addison-Wesley Publishing Company, Reading, Massachusetts, USA., pp: 299-311.
4. Sattinger, D.H. and O.L. Weaver, 1986. Lie Group and Algebras with Applications to Physics, Geometry and Mechanics, Springer-Berlag, New York, pp: 21-29, 119-127.
5. Cahen, M., S. Gutt, C. Kozameh and E.T. Newman, 1988. J. Math. Phys., 29: 1022-1025.
6. Helfer, A., M. Hickman, C. Kozameh, C. Lucey and E.T. Newman, 1987. Phys. Rev., 36: 1740-1744.
7. Bleecker, D., 1981. Gauge Theory and Variational Principles. Addison-Wesley Publishing Company, Inc. London, UK., pp: 55-63.