Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 6 (8): 1720-1724, 2006
ISSN 1812-5654

Action of Subgroups of G = <x,y; $x^2 = y^4 = 1$> on $Q^*(\sqrt{n})$

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Abstract: The study presents the action of subgroups of G = <x,y; $x^2 = y^4 = 1$> on $Q^*(\sqrt{n})$ and discuss some number-theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of subgroups $H_2 = \langle t,y: y^4 = 1 = t^2\rangle$, where $t = xy^2x$ and $H_3 = \langle t,y: y^4 = 1 = t^4\rangle$, where $t = xy^3x$, acting on $Q^*(\sqrt{n})$ and also compare our results obtained from H_3 with subgroup $H_1 = \langle t,y: t^4 = 1 = y^4\rangle$, where t = xyx.

Key words: Coset diagram, totally positive numbers, totally negative numbers, ambiguous numbers

INTRODUCTION

The group G is $\langle x, y:x^2 = 1 = y^4\rangle$, where $(\alpha)x = -1/2\alpha$, $(\alpha)y = -1/2(\alpha+1)$. The groups $H_1 = \langle t,y: t^4 = 1 = y^4\rangle$, where $(\alpha)t = (\alpha)xyx = 1-1/2\alpha$ and $(\alpha)y = -1/2(\alpha+1)$, $H_2 = \langle t,y:t^2 = 1 = y^4\rangle$, where $(\alpha)t = (\alpha)xy^2x = \alpha-1/2\alpha-1$ and $H_3 = \langle t,y:t^4 = 1 = y^4\rangle$, where $(\alpha)t = (\alpha)xy^3x = -1/2(\alpha-1)$ are thus subgroups of G.

Action of the subgroup $H_1 = \langle t,y:t^4 = 1 = y^4\rangle$, where t = xyx, of G on $Q^*(\sqrt{n}) = \left\{\frac{a+\sqrt{n}}{c}\right.$; a, cεZ; $b = (a^2-n)/c$ is rational integer and gcd (a, b, c) = 1} has been discussed (Aslam, 1997) which is in fact the main inspiration of our study.

In this study we are interested in the action of subgroups $H_2 = \langle t,y:t^2 = 1 = y^4\rangle$ and $H_3 = \langle t,y:t^4 = 1= y^4\rangle$ of $G = \langle x,y:x^2 = 1= y^4\rangle$ on $Q^*(\sqrt{n})$

We recollect that a quadratic irrational number $\alpha = \frac{a+\sqrt{n}}{c}$ has its algebraic conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ in $Q^*(\sqrt{n})$. If α and $\bar{\alpha}$ have different sign then α is called an ambiguous number. If they are both negative, then we call α is totally negative number and if α and $\bar{\alpha}$ both are positive then α is called totally positive number.

ACTION OF H_2 ON $Q^*(\sqrt{n})$

Let $a\in Q^*(\sqrt{n})$. The transformation t defined as $(\alpha)t = (\alpha)xy^2x = (\alpha-1)/(2\alpha-1)$ has order 2 and $(\alpha)y = -1/2(\alpha+1)$ has order 4, then for each α, $(\alpha)y$, $(\alpha)y^2$, $(\alpha)y^3$ and $(\alpha)y^4 = \alpha$, form vertices of a small square. If α is totally positive number, then all of $(\alpha)y$, $(\alpha)y^2$, $(\alpha)y^3$ are totally negative numbers (Mushtaq and Aslam, 1993). However, as we observe in our study that if α is totally negative number, then (α) t is totally positive number and when α is totally positive number, then (α) t may or may not be totally negative in general. Also we observe that if α is an ambiguous number, then (α) t is not ambiguous number. Before proving our result, we reproduce the following results (Kausar *et al*., 1997) which are being used in our discussions.

Lemma 1: (Kausar *et al*., 1997) An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally positive if and only if either a, b, c>0 or a, b, c<0

Lemma 2: (Kausar *et al*., 1997) An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number if and only if either c, b>0 and a<0 or c, b<0 and a>0

Lemma 3: (Kausar *et al*., 1997) An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number if and only if either c>0 and b<0 or c<0 and b>0

Theorem 4: (Mushtaq and Aslam, 1993) If $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally positive real quadratic irrational number, then $(\alpha)y^i$, for $1 \le i \le 3$, are totally negative numbers.

Theorem 5: If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, is totally negative real number, then (α) t is totally positive number.

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Proof: Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number. We are to prove that $(\alpha)t$ is totally positive number. For this, we consider

$$(\alpha)t = \frac{\alpha - 1}{2\alpha - 1} = \frac{(2b-3a+c)+\sqrt{n}}{4b-4a+c} = \frac{a_1+\sqrt{n}}{c_1} \quad \text{(say)}$$

where a_1 = 2b-3a+c, c_1 = 4b-4a+c and $b_1 = (a_1{}^2\text{-n})/c_1$ = $(2b\text{-}3a\text{+}c)^2$-n/4b-4a+c = $4b^2+8a^2+c^2$-12ab-6ac+4bc+a^2-n/4b-4a+c = $4b^2$+5bc-12ab-6ac+$8a^2+c^2$/4b-4a+c

By lemma 2, if $\alpha = \frac{a+\sqrt{n}}{c}$ is totally negative number, then either c, b>0 and a<0 or c, b<0 and a>0. This gives that
either

$a_1, b_1, c_1>0$ (In case of c, b>0 and a<0)

or

$a_1, b_1, c_1<0$ (In case of c, b<0 and a>0)

But, then, by lemma 1 (α) t is totally positive number. Hence the proof

Theorem 6: Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, be an ambiguous number, then (α) t is not an ambiguous number.

Proof: Note that if $\alpha = \frac{a+\sqrt{n}}{c}$ is an ambiguous number then, by definition either c>0, b<0 or c<0, b>0, that is bc<0. Thus b = $(a^2\text{-n})/c$, implies that $(a^2\text{-n})<0$. Which implies that a>0 or a<0. Using the calculations of Theorem 5, we have

a_1 = 2b-3a+c, c_1 = 4b+c-ra, $b_1 = a_1{}^2\text{-n}/c_1$
= $4b^2$+5bc-12ab-ac+$8a^2+c^2/c_1$

Following are the possible cases.

Case 1(a): a, b both are negative and c is positive (c>0, b<0, a<0)

Then -3a+c>0 and 2b<0, giving that a_1 = 2b-3a+c, either positive or negative. If $a_1>0$, then -3 a+c>2 b this implies that -4 a+c>4 b or 4 b+c-4 a>0. Since c_1 = 4 b-4 a+c therefore $c_1>0$. But we are not sure about b_1, whether it is positive or negative, we conclude that (α)t is not an ambiguous number. For $a_1<0$ we have that is -3 a+c<2 b. Since -4 a+c>0 and 4 b<0. Therefore either -4 a+c+4 b>0 or -4 a+c+4 b<0 that is $c_1>0$ or $c_1<0$. But we are not sure about b_1, whether it is positive or negative. So, we conclude that (α) t is not an ambiguous number.

Case 1(b): a, c both are positive and b is negative (c>0, b<0, a>0)
Here note that 2 b-3 a<0 and c>0 this implies that

2 b-3 a+c>0 or 2 b-3 a+c<0, that is a 1>0 or a 1<0

For $a_1>0$, we have $c_1>0$. But we are not sure about b_1, whether it is positive or negative. So we conclude that (α)t is not an ambiguous number.

Similarly for $a_1<0$, we have c<2 b-3 a which implies that c<4 b-4 a that is $c_1<0$. But this does not implies that $b_1>0$ or $b_1<0$. Therefore we conclude again that (α) t is not an ambiguous number.

Case 2(a): a, c are negative and b is positive (c<0, b>0, a<0)

In this case we note that 2 b-3 a>0 and c<0 which implies that 2 b-3 a+c>0 or 2 b-3 a+c<0. That is either a_1 is positive or a_1 is negative. For a<0 we have 4 b-4 a>c. That is $c_1>0$ and for $a_1<0$ we have 4 b-4 a+c>0 that is $c_1<0$ or $c_1>0$. But this does not implies that $b_1>0$ or $b_1<0$. Therefore we conclude that (α)t is not an ambiguous number.

Case 2(b): a, b are positive and c is negative (c<0, b>0, a>0)

We note that -3 a+c<0 and 2 b>0 which implies that 2 b-3 a+c>0 or 2 b-3 a+c<0. That is $a_1>0$ or $a_1<0$ for the case when $a_1>0$ we have 4 b>-4 a+c which implies that 4b-4 a+c>0. That is $c_1>0$, for $a_1<0$ we have 4 b-3 a+c<0 or 4b-3 a+c>0, that is $c_1<0$ or $c_1>0$. In both the cases we are sure, whether $b_1>0$ or $b_1<0$. Therefore we conclude that (α) t is not an ambiguous number.

We observe that (α) t is not an ambiguous number in all the cases. Hence the theorem is proved.

Theorem 7: Let $\alpha = \frac{a+\sqrt{n}}{c}$ be totally positive number.

- If (α) t is totally positive number, then either |2 b+c|>3 a, |4 b+c|>4 a or |2 b+c|<3 a, |4 b+c|<4 a and
- If (α) t is totally negative number, then either |2 b+c|>3 a, |4 b+c|<4 a or |2 b+c|<3 a, |4 b+c|>4 a

Proof: Let $\alpha = \frac{a+\sqrt{n}}{c}$ is totally positive number, then by lemma 1 either a, b, c>0 or a, b, c<0. Since we know that

(α) t = $(\alpha\text{-}1)/2\alpha\text{-}1$ = $\frac{a_1+\sqrt{n}}{c_1}$, where a_1 = 2 b+c-3 a,

c_1 = 4 b+c-4 a and $b_1 = (a_1{}^2\text{-n})/c_1$ = (4b2+5bc-12a-6ac+$8a^2+c^2$)/4b-4a+c

Suppose (α) t is totally positive number then, by lemma 1, either $a_1>0$, $b_1>0$, $c_1>0$ or $a_1<0$, $b_1<0$, $c_1<0$.

Now suppose that (α) t is totally negative then, by lemma 2, either $a_1>0$, $b_1<0$, $c_1<0$ or $a_1<0$, $b_1>0$, $c_1>0$

Now for the case of $a>0$, $b>0$, $c>0$ we observe that

$$a_1 = 2b+c-3a>0 \Rightarrow 2b+c>3a \quad (a)$$
$$a_1<0 \Rightarrow 2b+c<3a \quad (b)$$
$$b_1 = 4b+c-4a>0 \Rightarrow 4b+c>4a \quad (c)$$
$$b_1<0 \Rightarrow 4b+c<4a \quad (d)$$

and for $a<0$, $b<0$, $c<0$, we observe that

$$a_1 = 2b+c-3a>0 \Rightarrow 2b+c<-3a \quad (a')$$
$$a_1<0 \Rightarrow 2b+c>-3a \quad (b')$$
$$b_1 = 4b+c-4a>0 \Rightarrow 4b+c<-4a \quad (c')$$
$$b_1<0 \Rightarrow 4b+c>-4a \quad (d')$$

From (a) and (a′) we see that $a_1>0$ only when $3a<2b+c<-3a$ that is

$$|2b+c|>3a \quad (e)$$

From (b) and (b′) we see that $a_1<0$ only when $-3a<2b+c<3a$ that is

$$|2b+c|<3a \quad (f)$$

From (c) and (c) we see that $b_1>0$ only when $4a<4b+c<-4a$ that is

$$|4b+c|>4a \quad (g)$$

From (d) and (d) we see that $b_1<0$ only when $-4a<4b+c<4a$ that is

$$|4b+c|<4a \quad (h)$$

When (α) t is totally positive we have either

$|2b+c|>3a$, $|4b+c|>4a$ (In case of $a_1>0$, $b_1>0$, $c_1>0$)
or
$|2b+c|<3a$, $|4b+c|<4a$ (In case of $a_1<0$, $b_1<0$, $c_1<0$) (A)

When (α) t is totally negative we have either

$|2b+c|>3a$, $|4b+c|<4a$ (In case of $a_1>0$, $b_1<0$, $c_1<0$)
or
$|2b+c|<3a$, $|4b+c|>4a$ (In case of $a_1<0$, $b_1>0$, $c_1>0$) (B)

From (A) and (B) theorem is proved.

Coset diagram of H_2 = <t, y ; $t^2 = y^4 = 1$>where $t = x y^2 x$: It has been shown (Mushtaq and Aslam, 1993) while studying the action of $G = \langle x, y: x^2 = 1 = y^4\rangle$ on $Q^*(\sqrt{n})$ if α is an ambiguous vertex of a square in coset diagram for α^G then $(\alpha)x$ is an ambiguous number and one of the vertices $(\alpha)y$, $(\alpha)y^2$, $(\alpha)y^3$ is ambiguous and other two are totally negative. In other words each ambiguous number is joined by x-edge or by y-edge, to other two ambiguous numbers. Since there are finite ambiguous numbers in $Q^*(\sqrt{n})$ (Mushtaq, 1983), therefore ambiguous vertices form a closed path in the Coset diagram.

In our case we have proved that if α is an ambiguous number, then (α) t is not ambiguous number (Theorem 6). Hence each square which have two vertices ambiguous, will not form the closed path in the diagram. But then if α is totally negative number then (α) t is totally positive number (Theorem 5), therefore the general fragment of coset diagram will be as shown in Fig. 1.

Action of H_3 = <t, y ; $t^4 = y^4 = 1$>on $Q^*(\sqrt{n})$: We have discussed some number-theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of subgroup $H_3 = \langle t, y: y^4 = 1 = t^4\rangle$, where $t = xy^3x$, acting on $Q^*(\sqrt{n})$ and then compare our results obtained from H_3 with subgroup $H_1 = \langle t, y: t^4 = 1 = y^4\rangle$, where $t = xyx$, which was discussed Aslam (1997) already. The results we have proved are;

Theorem 8: If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative then $(\alpha)t^i$ is totally positive for each i = 1, 2 or 3.

Proof: If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, then $(\alpha)t = -1/2(a-1) = a_1 - \sqrt{n}/c_1$ where $a_1 = -a+c$, $b_1 = c/2$, $c_1 = -4a+2b+2$

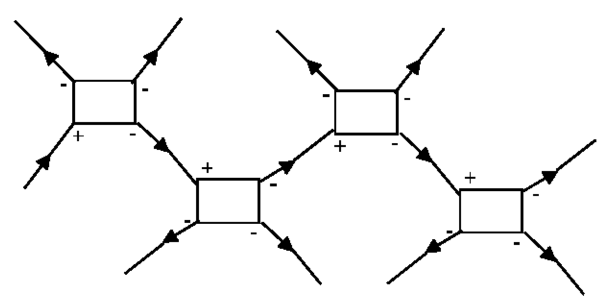

Fig. 1: Coset Diagram of H_2, in which each square have two vertices ambiguous, will not form the closed path since (α) t is not ambiguous number for a an ambiguous number

are new values of a, b, c respectively. Similarly in case of $(\alpha)t^2 = 1-\alpha/1-2\alpha$ and $(\alpha)t^3 = 1-1/2\alpha$, the new values of a, b, c are $a_2 = -3a+2b+c$, $b_2 = -2a+b+c$, $c_2 = -4a+4b+c$ and $a_3 = -a+2b$, $b_3 = -4a+4b+c/2$, $c_3 = 2b$, respectively.

If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number then by lemma 2, either a<0 and b>0, c>0 or a>0 and b<0, c<0. When a<0 and b>0, c>0, we see that (for each case, $(\alpha)t^i$, i = 1, 2 or 3), the new values of a, b and c are positive. Hence by lemma 1 $(\alpha)t^i$ are totally positive. Similarly, when a>0 and b<0, c<0, we observe that the new values of a, b and c for each $(\alpha)t^i$ i = 1, 2 or 3, are negative. Therefore by lemma 1 each $(\alpha)t^i$ i = 1, 2 or 3 is totally positive. Hence the proof

Theorem 9: If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number then, for i = 1, 2, or 3, one of, $(\alpha)t^i$ is an ambiguous number and other two are totally positive.

Proof: (Case 1) Let α is negative number then possibilities for signs of $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$, $\overline{(\alpha)t}$, $\overline{(\alpha)t^2}$ and $\overline{(\alpha)t^3}$ are given in Table 1.

(Case 2) Now consider the case when α is positive number. In this case the possibilities for signs of $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$, $\overline{(\alpha)t}$, $\overline{(\alpha)t^2}$ and $\overline{(\alpha)t^3}$ are given in Table 2.

Therefore from the cases ((Case 1) and (Case 2)) above, we deduce that for i = 1, 2 or 3, one of $(\alpha)t^i$ is an ambiguous number and other two are totally positive. Hence the proof

The coset diagram (Fig. 2) can also illustrate proof of the above theorem:

Further it has been observed (Mushtaq and Aslam, 1993) that if k ≠ 0, -1/2, -1 or ∞ is one of the four vertices of a square in a coset diagram, then

- z<-1 implies that (z) y>0
- z>0 implies that -1/2<(z)y<0
- -1/2<z<0 implies that -1<(z)y<-1/2
- -1<z<-1/2 implies that z<-1

that is if the vertices k, k y, k y^2, k y^3 of a square are not 0, -1/2, -1 or ∞, then one of these four vertices is positive and the other three are negative.

It have been studied by Aslam (1997) that if if k ≠ 0, 1/2, 1 or ∞ is vertex of a square in Coset diagram, then

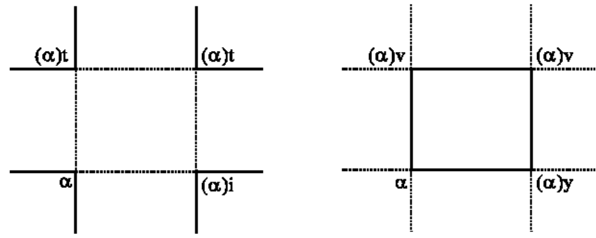

Fig. 2: Coset Diagram of H_3 in which 4 cycles of y are defined by four edges (unbroken) and 4 cycles of the transformation t = x y^3 x are defined by four edges (broken) of a square, permutated counter clockwise by both y and t

Table 1: Possibilities for sign of $\overline{(\alpha)t^i}$ (for i = 1, 2, 3 and 4) when α is negative

α	(α)t	$(\alpha)t^2$	$(\alpha)t^3$	$\bar{\alpha}$	$\overline{(\alpha)t}$	$\overline{(\alpha)t^2}$	$\overline{(\alpha)t^3}$
-	+	+	+	+	-	+	+
				+	+	-	+
				+	+	+	-

Table 2: Possibilities for sign of $\overline{(\alpha)t^i}$ (for i = 1, 2, 3 and 4) when α is positive

α	(α)t	$(\alpha)t^2$	$(\alpha)t^3$	$\bar{\alpha}$	$\overline{(\alpha)t}$	$\overline{(\alpha)t^2}$	$\overline{(\alpha)t^3}$
+	-	+	+	-	+	+	+
+	+	-	+				
+	+	+	-				

- z<0 implies that (z)t>1
- z>1 implies that 1/2<(z)t<1
- 1/2<z<1 implies that 0<(z)t<1/2
- 0<z<1/2 implies that (z)t<0

that is if vertices k, k t, k t^2, k t^3 of a square are not or then one of four vertices is negative and other three are positive.

In our case, we study that if k≠0, 1/2, 1 or ∞ is vertex of a square in Coset diagram, then

- z<0 implies that 0<(z)t<1/2
- 0<z<1/2 implies that 1/2<(z)t<1
- 1/2<z<1 implies that (z)t>1
- z<1 implies that (z)t<0

that is if vertices k, k t, k t^2, k t^3 of a square are not 0, 1/2, or ∞ then one of four vertices is negative and other three are positive but with different values discussed by Aslam (1997). Therefore, the number-theoretic properties belonging to the orbit of subgroup $H_3 = \langle t,y: y^4 = 1 = t^4\rangle$, where $t = xy^3x$, are same as number-theoretic properties in case of subgroup H_1

The following coset diagram (In which 4-cycles of y are defined by four edges (unbroken) and 4-cycles of the

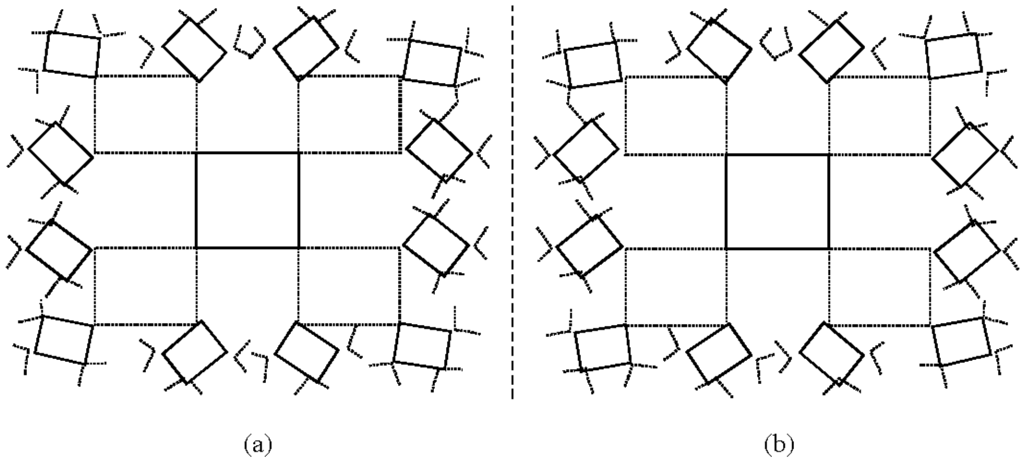

Fig. 3(a): Fragment of Coset diagram for H_1
(b): Fragment of Coset diagram for H_3

transformation $t = xy^3x$ are defined by four edges (broken) of a square, permutated counter clockwise by both y and t. Fixed points by y and t are denoted by heavy dots) summarize the whole discussion. The vertices of 4 cycles of transformations $t = x y^3 x$ are permutated in the opposite direction to the vertices of t = x y x. The general fragments of Coset diagrams for $H_1 = \langle t, y; t^4 = y^4 = 1\rangle$ where t = xyx and $H_3 = \langle t, y; t^4 = y^4 = 1\rangle$ where $t = x y^3 x$, are illustrated in the Fig. 3(a, b).

Note that the Fig. 3(a) and 3(b) are reflections of each other.

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