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Statistically Improved Approximation By Modified Lupas Operator

[1]Ashok Sahai, [1]Angela Shirley and [2]M. Raghunadh Acharya
[1]Department of Mathematics and Computer Science,
The University of the West Indies, St. Augustine, Trinidad and Tobago
[2]Department of Statistics and Computer Science, Aurora's Post Graduate College,
Osmania University, Hyderabad Andhra Pradesh, India

Abstract: In 1950, Szasz proposed a generalization of the well-known Bernstein's polynomials extending it to the infinite interval. Many authors such as, Hermann in 1977, studied its use in the approximations of functions on an unbounded interval. The actual construction of the Szasz-Mirakjan Operator, say $S_n(f;x)$, requires estimation per infinite series, which apparently restricts its practical usefulness from the computational point-of-view. In 1980, Grof introduced 'Modified Szasz-Mirakjan Operator' which was a finite 'partial sum' curtailment of $S_n(f;x)$ and studied it. In 1984, Heinz-Gerd Lehnhoff, in particular, proposed for f and x ε C [0, 1] another 'Modified Szasz-Mirakjan Operator': $S_n(f;x) = [e^{-nx}\sum_{k=0}^{k=n}\{(nx)^k/k!\}.f(k/n)]$. We have proposed and studied a statistically motivated improvement of the Lupas Operator modified analogously. The study is supported and illustrated by the following empirical simulation study aimed at bringing forth the potential improvement numerically for some standard types of function.

Key words: Polynomial approximation, szasz-mirakjan operator, lupas operator, simulated empirical study

INTRODUCTION

There has been an intermittent interest in polynomial approximation following Weierstrass (1885). For the readers not much aware of polynomial approximation, it might be a desirable orientation for this study to be perused with the references: Carothers (1998 and 2000), Grof (1980), Hermann (1977), Hedrick (1927), Lorentz (1986) and Weierstrass (1885). For the similar orientation for a numerical analysis and computing context, helpful references would be Cheney and Kincaid (1994), Hartley and Wynn-Evans (1979) and Polybon (1992). In the context of this study Szasz (1950) proposed the following generalization of the well-known Bernstein's polynomials extending it to the infinite interval:

$$S_n(f;x) = [e^{-nx}\sum_{k=0}^{\infty}\{(nx)^k/k!\}f(k/n)], \text{ for all } f \in C_A[0,\infty]$$

Heinz-Gerd (1984), in particular, proposed Modified Szasz-Mirakjan Operator'

$$S_n(f;x) = [\sum_{k=0}^{n} T_k f(k/n)]/[\exp(nx)], \text{for } f \in C[0,1], x \in C[0,1]$$

Where $T_k = (nx)^k/k!$, $\forall$ k = 0,...,n.

Several Modifications of Lupas Operator have been proposed and studied, e.g. by Sahai and Prasad (1985). Motivated by the above modification, we have proposed analogously, though slightly differently, a Modified Lupas Operator', as follows:

$$ML[n] \equiv \sum_{k=0}^{n}\binom{n+k-1}{k}x^k f(k/(k+n)/\sum_{k=0}^{n}\binom{n+k-1}{k}x^k \qquad (1)$$

The aforesaid 'Modified Lupas Operator' ML[n] will approximate the function f (x) using its values at the 'knots' in the interval [0, 0.5]. Therefore, if we approximate the function f (2x) in the interval [0, 0.5], virtually we would be able to approximate the function f (x) in the interval [0,1].

Further, we have proposed and studied, in what follows, a computerizable "Iterative Algorithm" with the motivation of having improved approximation by the aforesaid operator "ML[n]", using the same 'information', namely, the values of the function at the stipulated 'knots'.

Corresponding Author: Angela Shirley, Department of Mathematics and Computer Science, The University of the West Indies, St. Augustine, Trinidad and Tobago

MOTIVATING OBSERVATION AND THE ITERATIVE ALGORITHM

Before we take to the detailing of the 'Iterative Algorithm' for improved approximation by our 'Modified Lupas Operator, ML[n]', we observe the motivating fact seminal to its proposition. Whereas, all the approximating polynomials are concerned with the Knots and the *weight functions* defined over these knots; none of them completely uses the information available about the unknown function (targeted for the Approximation), through the known values of the function at these knots. Such information could well be used in constructing/modifying the weight function, possibly gainfully.

In fact, as per the Statistical Perspective, such an information should be used gainfully in all the Estimation Problems; And the Approximation Problem is an estimation problem per this perspective, as we are essentially estimating the unknown function through our weight function, defined at the chosen Knots for the approximation operator, at hand.

In fact, if we particularly confine to the polynomial approximation by Positive Linear Operators, we could well observe the fact that the weights being interpretable as probabilities, in the context of using the Operator, say $O_n(f)(x)$, the desirable/well known Statistical Property of Asymptotic Unbiasedness ensures that the Mathematical Expectation (the value 'on an average') of our approximating polynomial, namely the estimate $O_n(f)(x)$ must approach the function, as the number of knots used, namely n becomes very large: $E\{O_n(f)(x)\} \to f(x)$, as $n \to \infty$.

In the above context and using the aforesaid Statistical Perspective of the approximation being an estimation problem, the estimated error could well be interpreted as the estimated *Bias*. As such, therefore, if we adjust this bias, to keep the estimate better, we are not only ensuring the asymptotic unbiasedness of the estimate, but we are eventually accelerating the asymptotic convergence of the approximating polynomial. This very desirable attempt has been *seminal* to the proposition of our Iterative Algorithm, targeting at improving the approximating polynomial.

This will be feasible, inasmuch as we would be reducing the Error in approximation at each *iteration*, using the currently available estimate of the error to bring the approximating polynomial *closer* to the (unknown) function.

Now, we denote Error by E and we have:

$$E(x) = ML[n](f)(x) - f(x).$$

However, E(x) is unknown inasmuch as f(x) is so. Therefore, essentially we have to estimate it and we do so by using the same Modified Lupas polynomial ML[n](f)(x). The only difference, apparently, would consist in the fact that we have E in place of f and analogously the values of this unknown Error Function E(x) are readily available through the difference between the Known and the *Estimated* values of the function at these knots: (k/n), respectively.

Hence, if we define the resultant estimating polynomial (apparently of degree n, at the most) by E[n](f)(x), (keeping in mind implicitly that ML[n](f)(x) is the approximating polynomial, without complicating the notations by explicit incorporation of this fact in our notation), we have:

$$\begin{aligned} E[n](f)(x) &\equiv ML[n]\{ML[n](f)-f\}(x) \\ &\equiv ML[n]^2(f)(x) - ML[n](f)(x). \end{aligned}$$

Also, as we use this polynomial as an Estimated Bias and proceed with the correction, the resultant Improved Lupas approximating polynomial, at the first go/iteration, say I(1) ML[n](f)(x) will be:

$$\begin{aligned} I(1)\, ML[n](f)(x) &= ML[n](f)(x) - E[n](f)(x) \\ &= 2\, ML[n](f)(x) - ML[n]^2(f)(x) \\ &= [I - (I - ML[n])^2(f)(x) \qquad (2) \end{aligned}$$

Apparently, if we proceed exactly analogously for the Improved Lupas' Approximating polynomial at the second iteration, we will be led to

$$\begin{aligned} I(2)\, ML[n](f)(x) &= I(1)\, ML[n](f)(x) \\ &\quad - ML[n] I(1)\ \{ML[n](f)(x) - (f)(x)\} \\ &= 2\, ML[n](f)(x) - ML[n]^2(f)(x) \\ &\quad -2ML[n]^2(f)(x) + ML[n]^3(f)(x) \\ &\quad +ML[n](f)(x) \\ &= 3\, ML[n](f)(x) - 3\, ML[n]^2(f)(x) \\ &\quad +ML[n]^3(f)(x) = [I - (I - ML[n])^3(f)(x) \qquad (3) \end{aligned}$$

Thus, in general, if we proceed exactly analogously for the Improved Lupas approximating polynomial at the kth iteration, we will be led to:

$$\begin{aligned} I(k)\, ML[n](f)(x) &= [I - (I - ML[n]^{k+1})](f)(x), \\ k &= 0(1).... \qquad (4) \end{aligned}$$

Now, we note that whereas it is intractable analytically to assess the achieved improvement in Approximation by the Modified Lupas Operator ML[n] through the aforesaid Iterative Improvement Algorithm, we resort to an Empirical Simulation Study to have the numerical flavor of the goodness of the Algorithm, as in the following section. It could well be noted that the Algorithm is evidently Computerizable for its execution.

THE EMPIRICAL SIMULATION STUDY

To illustrate the gain in efficiency by using our proposed Iterative Algorithm of Improvement of Modified Lupas Polynomial Approximation, we have carried out an Empirical Study. We have taken the example-cases of n = 2, 4, 7 and 10, (i.e., n + 1 = 3, 5, 8 and 11 as knots) in the empirical study to numerically illustrate the relative gain in efficiency in using the Algorithm vis-à-vis the Modified Lupas Polynomial in each example- case of the n-value.

Essentially, the empirical study is a Simulation Empirical Study, because we would have to assume that the function being tried to be approximated, namely f(x) (by taking f(2x), as detailed in the first section in the paper: the reason being that it will then be approximating the function f(x) in the interval [0,1], {the standard 'Conventional' interval for the 'Approximation of the function f(x)'})being known to us.

Once again, we have confined to the illustrations of the relative gain in efficiency by the Iterative Improvement to the approximation of the following four illustrative functions in the interval [0, 1]:

$$f(x) = \exp(x),\ \ln(2+x),\ \sin(2+x),\ 10^x$$

So that for our purposes, as detailed in the 'First' section of the paper our target functions in the proposed "Empirical Simulation Study" would be:

$$f(2x) = \exp(2x),\ \ln(2+2x),\ \sin(2+2x),\ 10^{2x}$$

These would be approximated in the interval [0, 0.5] by the Modified Lupas Operator ML[n] and subsequently also approximated by using the Computerizable Iterative Improvement Algorithm

To illustrate the POTENTIAL of improvement with our proposed Iterative Algorithm, we have considered THREE Iterations and the numerical values of four quantities, namely three Percentage Relative Errors (PRE) corresponding to our Improvement Iterations (# = 1, or 2, or 3), (PRE_I(#)ML[n]) and the Modified Lupas Polynomial (PRE_ML[n]). Also, we consider the corresponding three Percentage Relative Gains (PRG) by using our Proposed Iterative Algorithmic Modified Lupas Polynomial subsequent upon the approximation by Modified Lupas Polynomial, (PRG_I(#)ML[n]; # = 1(1)3). Now, these quantities are defined, as follows.

The Percentage Relative Error using Modified Lupas polynomial with n intervals in [0, 0.5], i.e., [(k-1)/(k+n-1), k/(k+n)];k = 1(1)n:

$$\mathrm{PRE\_ML}[n] = \frac{\left|\int_0^1 f(x)dx - \int_0^1 \mathrm{ML}[n](f)(x)dx\right|}{\int_0^1 f(x)dx}\times 100$$

The Percentage Relative Error using Improvement Iteration; I #: 1, or 2, or 3 on Modified Lupas Polynomial using n intervals in [0,0.5], i.e. [(k-1)/(k+n-1), k/(k+n)]; k = 1(1)n:

$$\mathrm{PRE\_I(\#)ML}[n] = \frac{\left|\int_0^1 f(x)dx - \int_0^1 \mathrm{I(\#)ML}[n](f)(x)dx\right|}{\int_0^1 f(x)dx}\times 100;$$

= 1, or 2 or 3

The Percentage Relative Errors respective to the Modified Lupas Polynomial and respective to the First, Second and the Third Algorithmic Improvement Iteration Polynomials have been tabulated respectively, for each of the examples and the number of approximation Knots/Intervals. Also the Percentage Relative Gains by using the proposed Algorithmic Improvement Iteration: *I#* (e.g. 1, or 2, or 3) Polynomials with the n intervals in [0,0.5] over using solely the Modified Lupas Polynomial for the approximation of the (targeted) function, f, are given in the Table (1-4).

Table 1: Algorithmic improvement efficiency [f(x) = exp(2x)]

Model ↓ / n →	2	4	7	10
PRE_ML[n]	15.718293	9.948736	6.537999	4.906638
PRE_I(1)ML[n]	6.943140	4.169742	2.602362	1.894311
PRE_I(2)ML[n]	5.769793	3.096123	1.831332	1.311473
PRE_I(3)ML[n]	5.611988	2.483332	1.491115	1.058251
PRG_I(1)ML[n]	55.827644	58.087722	60.196350	61.392862
PRG_I(2)ML[n]	63.292493	68.879235	71.989408	73.271436
PRG_I(3)ML[n]	64.296452	75.083714	77.193106	78.432242

Table 2: Algorithmic improvement efficiency [f(x) = sin(2+2x)]

Model ↓ \ n →	2	4	7	10
PRE_ML[n]	22.281402	14.074188	9.269027	6.947331
PRE_I(1)ML[n]	10.053345	5.849850	3.388898	2.329825
PRE_I(2)ML[n]	7.910855	4.157418	2.157609	1.428652
PRE_I(3)ML[n]	7.575087	3.120199	1.562034	1.103813
PRG_I(1)ML[n]	54.880106	58.435612	63.438490	66.464455
PRG_I(2)ML[n]	64.495705	70.460689	76.722383	79.435963
PRG_I(3)ML[n]	66.002647	77.830342	83.147814	84.111693

Table 3: Algorithmic improvement efficiency [f(x) = ln(2+2x)]

Model ↓ \ n →	2	4	7	10
PRE_ML[n]	10.339364	6.108482	3.762205	2.718718
PRE_I(1)ML[n]	4.333963	1.541269	0.605614	0.413721
PRE_I(2)ML[n]	1.669836	0.508464	0.377975	0.294215
PRE_I(3)ML[n]	0.666215	0.424826	0.322721	0.235773
PRG_I(1)ML[n]	58.082882	74.768375	83.902698	84.782484
PRG_I(2)ML[n]	83.849719	91.676101	89.953352	89.178170
PRG_I(3)ML[n]	93.556521	93.045316	91.421747	91.327775

Table 4: Algorithmic improvement efficiency [f(x) = 10^{2x}]

Model ↓ \ n →	2	4	7	10
PRE_ML[n]	27.062782	20.03825	14.809442	11.890574
PRE_I(1)ML[n]	20.109165	13.322666	8.853231	6.639897
PRE_I(2)ML[n]	19.713509	10.512656	6.616282	4.837491
PRE_I(3)ML[n]	19.051423	8.919115	5.588074	3.418436
PRG_I(1)ML[n]	25.694391	33.513807	40.219013	44.158311
PRG_I(2)ML[n]	27.156384	47.537041	55.323897	59.316592
PRG_I(3)ML[n]	29.602866	55.489542	62.266820	71.250871

$$PRG\_I(\#)ML[n] = \frac{PRE\_ML[n] - PRE\_I(\#)ML[n]}{PRE\_ML[n]} \times 100;$$

$\# = 1$, or 2 or 3

CONCLUSIONS

These aforesaid Seven numerical quantities have been computed using Maple V Release 3, for all the four illustrative functions mentioned in the preceding Section 3, for four values of n, namely n = 2,4,7 and 10. These values have been given in Table (1-4). Table 1-4 contain these quantities when the function f(x) has been taken as exp(2x), ln(2+2x), sin(2+2x) and 10^{2x}, respectively.

The Percentage Relative Error (PRE's) for our Algorithmic Iterative Polynomial Approximations are PROGRESSIVELY lower with each subsequent iteration, as compared to that for the Modified Lupas Polynomial Approximation, for all the illustrative functions.

Consequently, the Percentage Relative Gains (PRG's) due to the use of our proposed Algorithmic Iterative Polynomial Approximations vis-à-vis the Modified Lupas Polynomial Approximation are also increasing PROGRESSIVELY with each subsequent iteration, for all the illustrative functions.

Lastly, it is very heartening to note that when we use TEN (n = 10) intervals, i.e., ELEVEN KNOTS for polynomial approximation, Percentage Relative Gain (PRG) becomes quite significant for third iteration. Otherwise also, the speed of convergence is highly accelerated by the Iterative Algorithmic improvement in the Modified Lupas Polynomial, using the statistical perspective reducing the Bias in the Estimator/Approximating Polynomial. It is worth noting again that the Modified Lupas Operator is nothing but the weighted average of the data, i.e. the known values of the unknown function f at the n +1 knots.

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