Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 6 (8): 1854-1857, 2006
ISSN 1812-5654

Approximation of Fixed Points of Certain Linear Pseudocontractive Map by a Stochastic Iterative Method

A.C. Okoroafor and B.O. Osu
Department of Mathematics, Abia State University, Uturu

Abstract: Let $S = \{x \in R^n: x>0, x'x \leq r^2\}$ be a compact subset of R^n and let $T: D(T) \subseteq S$ with domain D(T) be positive linear pseudocontractive map induced by a positive irreducible matrix. We establish, the strong convergence of a recursive stochastic approximation algorithm of Robbins-Monroe type to the unique fixed point of T.

Key words: Linear pseudocontractive maps, stochastic fixed point iteration, irreducible matrix

INTRODUCTION

Let R^n be n-dimensional Euclidean space. If T and $T^*$are mappings with domains

$$D(T) \text{ and } D(T^*) \text{ in } R^n$$

and with values in R^n, then such T is called strongly Pseudocontractive if there exist $\alpha<1$ such that the inequality

$$\langle Tx-Ty,x-y\rangle<\alpha\|x-y\|^2 \quad (1)$$

holds for all $x,y \in D(T)$ while T^* is said to be strongly accretive if

$$\langle T^*x-T^*y,x-y\rangle \geq (1-\alpha)\|x-y\|^2 \quad (2)$$

Let M be n x n matrix with positive elements

$$a_{ij} \text{ and } \min\sum_i^n a_{ij} < r < \max\sum_i^n a_{ij} \quad r>0$$

We assume that the iterates

$$M^k = \{a_{ij}^k\} \text{ of } M$$

are defined and finite and that for each pair of indices i,j there exists an integer t>0 depending in general on i and j such that

$$a_{ij}^t > 0 \text{ (that is, M is irreducible)}$$

We define

$$Tx = Mx \quad (3)$$

where M is considered as an operator acting on column vectors having positive entries.

Then elementary calculation yields

$$\langle T^*x - T^*y, x-y\rangle \leq \alpha\|x-y\|^2 \text{ for all } \alpha = \sum_{i=1}^n a_{ij} \leq 1 \quad (4)$$

so that clearly, T is a pseudocontractive mapping. Hence by the fixed point theorem of Browder (1967) there is a unique fixed point $x_0 \in S$ with $x_0 = Tx_0$. A number of results recurringly used in Economics concern the theorem about sI-T, s>0, where I is identity matrix (Kemeny and Snell, 1960; Solow, 1952). For instance, a well known result relevant in simple direct discussion of linear models of production in econometrics is that the equation (sI-T)x = y has a unique positive solution given by $x = (sI-T)^{-1}y$ for every y, s>r (Solow, 1952).

Sometimes the condition $Tx \leq x$ is imposed as a fundamental assumption so that there exist a $y \geq 0$ such that Tx = x+c. This assumption is relevant to the productivity in linear models (Gale, 1960). It is shown in Pruitt (1964) that if M is irreducible and $sMx \leq x$ for some s>0 and non trivial x, then $x_j>0\ \forall$ j.

Thus if s = 1, the problem of finding x* such that

$$(I-T)x^* = \min_{x\in D(T)}(I-T)x = 0 \quad (5)$$

is equivalent to finding $x^*>0$ such that $Tx^* = x^*$.

The firm connection between the pseudocontractive mappings and the accretive operators is that a mapping T is pseudocontractive if and only if I-T is accretive (Browder, 1967; Chidume, 1987) so that any solution of the equation $T^*x = 0$ is also a solution of Tx = x.

This leads us to consider a stochastic sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \rho_n d_n \quad (6)$$

Corresponding Author: B.O. Osu, Department of Mathematics, Abia State University, Uturu

whose limit is the solution of the equation $T^*x = 0$ where $T^* = 1-T$.

Chidume (1987) has proved the

Theorem 1: Suppose S a nonempty closed bounded and convex subset of L_p $p \geq 2$ and $T: S \to S$ is a lipschitz strongly pseudocontractive mapping of S into itself.

Let $\{\rho^k\}$ be a real sequence satisfying:

$$(1)\quad 0 < \rho^k < 1$$

$$(2)\quad \sum_{k=1}^{\infty} \rho^k = \infty$$

$$(3)\quad \sum_{k=0}^{n} \rho^{2k} < \infty$$

Then the sequence $\{x^k\}_{k=0}^{\infty}$ generated by $x^{/} \in S$, $x^{k+1} = x^k - \rho^k T^* x^k$, $k \geq 1$ converges strongly to a solution of the equation $T^*x = 0$ where $T^* = I - T$.

The main concern of this study is to prove strong convergence with probability one of the stochastic sequence to the fixed point of T.

PRELIMINARIES

We proceed now with some background in notation and definitions. Here, as in the sequel, $\| x \|$ denotes the Euclidean norm of $x \in R^n$ and given the points x and y in R^n, $\langle x, y \rangle$ is the usual inner product in R^n.

For the purpose of this study a random variable will be a numerically valued observable X in a probability space (Ω, f, P). The probability measure P is characterized by the singleton probabilities

$$p_j = P(\{\omega_j\}), \text{where} \{\omega_j\} = \{\omega_1, \omega_2, ..., \omega_n\} \text{ which satisfy}$$

$$p_j \geq 0, \sum_{j=1}^{n} p_j = 1 \qquad (7)$$

Let E denote the expectation operator. The expectation of the observation is by definition,

$$EX = \sum_{\lambda = R} P(x = \lambda) = \sum_{j=1}^{n} p_j X(\omega_j) \qquad (8)$$

If $E\|x\| < \infty$ then Ez is defined by the requirement that $E(a,z) = (a,Ez)$ for the random element z and a fixed a in R^n, where $\|z\|, \langle z_1, z_2 \rangle, \langle a, z \rangle$ are real valued observable random variables in the usual sense.

In order to solve the problem by successive approximation it is convenient to transform the equation to the form

$$f(x) = \frac{x'Tx}{2} \qquad (9)$$

so that $\partial f(x)/\partial x = Tx$ and since T is positive definite, there exists a minimum of f in R^n and every minimizing sequence coverages to the unique minimum of f at

$$\left\{ x^* : \frac{\partial f(x^*)}{\partial x} = 0 \right\} \qquad (10)$$

We shall now obtain the estimate of the gradient mapping

$$\frac{\partial f(x)}{\partial x} \qquad (11)$$

using the following approximation theory

APPROXIMATION THEORY

We assume that for each set of values $x_1, x_2, ..., x_n$, using functions values $y(x_1)$, $y(x_2)$, $....y(x)_n$ as observations, there are uncertainties associated with these observations

$$y(x_j),\ j = 1, ..., m \qquad (12)$$

This can be modelled by adding a white noise process

$$e(x_j),\ j = 1, ..., m \qquad (13)$$

which are independently distributed about zero with common variance σ^2.

The resulting model for the stochastic process is then given by

$$f(x^* + t_j) - f(x^*) = y(x_j) = \left\langle \frac{\partial f(x^*)}{\partial x}, t_j \right\rangle + e(x_j) \qquad (14)$$

for a giving $t_j \in R^n$. From this it follows that if f is transformed to a linear function of x with coefficients $\frac{\partial f(x)}{\partial x}$ the most efficient method of fitting is the method of least square, which yields un biased estimate

$$d \text{ of } \frac{\partial f(x)}{\partial x}$$

having the least possible variance.

The natural Taylor's expansion of a quadratic function of f about a point X^* is given by

$$f(x)-f(x^*)=\left\langle \frac{\partial f(x^*)}{\partial x}, x-x^* \right\rangle + \frac{1}{2}(x-x^*)H(x_c)(x-x^*) \quad (15)$$

where x_c is on the line segment between x and x_c and $H(x_c)$ is the Hessian of f at x_c so that for a fixed k and j = 1,…, m.

If $y(x_1)$, $y(x_2)$, ..., $y(x_m)$ are real-valued independent observable random variables performed on x_1, x_2, ..., x_m chosen in the neighborhood of a fixed x^k

then

$$y(x_j)=f(x^*+t_j)-f(x_j)=\left\langle \frac{\partial f(x^k}{\partial x}, t_j \right\rangle + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} t_{ij}t_{kj}\frac{\partial^2 f(x)}{\partial x_i \partial x_k} + e(x_j) \quad (16)$$

is identifiable with (15).

$$\text{Let } Q=\left\{ \begin{array}{l} Ed^k:\left(\sum t_j t_j'\right)d^k=\sum t_j y(x_j), \\ y(x_j)=\left\langle \frac{\partial f}{\partial x}, t_j \right\rangle + e(x_j) \end{array} \right\} \quad (17)$$

and

$$\varphi=\left\{ Eu^k:\left(\sum t_j t_j'\right)u^k=\sum \begin{array}{l} t_j y_j, y_j=\left\langle \frac{\partial f(x^*)}{\partial x}, t_j \right\rangle + \\ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} t_{ij}t_{kj}\frac{\partial^2 f(x)}{\partial x_i \partial x_k} + e(x_j) \end{array} \right\} \quad (18)$$

Then for a fixed t_j in R^n satisfying

$$\sum_{j=1}^{m} t_{ij}=0 \text{ and } \frac{1}{m}\sum_{j=1}^{m} t_{ij}^2=1, i=1,2,...,n \text{ (Whittle, 1976)} \quad (19)$$

$$Ed^k=\left(\sum t_j t_j'\right)^{-1}\sum t_j E(y_j)=\frac{\partial f}{\partial x}$$

and

$$Eu^k=\left(\sum t_j t_j'\right)^{-1}\sum t_j E(y_j) = \frac{\partial f}{\partial x}.$$

An easy calculation yields

$$E\left\| d^k-\frac{\partial f(x^k)}{\partial x} \right\|=0 \text{ and } E\left\| d^k-\frac{\partial f(x^k)}{\partial x} \right\|^2=\left(\sum t_j t_j'\right)^{-1}\sigma^2<\infty \quad (20)$$

Thus by the condition (19), tj linearizes f so that the relationship between y_j and t_j, $t_j = x_j\text{-}x^k$ is adequately approximated by the model

$$y_j=\left\langle \frac{\partial f}{\partial x}, t_j \right\rangle + e_j \quad (21)$$

and the least squares approximation

$$d^k=M^{-1}\sum t_j y_j, M=\sum t_j t_j' \quad (22)$$

exists and is adequate for approximating

$$\frac{\partial f}{\partial x}$$

such that

$$E\left\| d^k-\frac{\partial f}{\partial x} \right\|=0$$

for each k and yields a minimum Euclidean distance between the true and the estimated gradient vector

$$E\left\| d^k-\frac{\partial f}{\partial x} \right\|^2$$

The stochastic gradient type recursive sequence

$\{x^k\}$, defined by $x^{k+1} = x^k\text{-}\rho^k d^k$

is suggested, where $\{\rho^k\}$ is a sequence of positive scalar to be specified and $\{d^k\}$ is a sequence of independent and identically distributed random vectors.

This procedure is a way of stochastically solving the equation

$$\left\{ x^*:\frac{\partial f(x^*)}{\partial x}=0 \right\}$$

The stochastic algorithm is thus given as follows:
Let x^k be giving an estimate of x^*

(a) compute $\partial f \cong d$ as in (22) earlier
(b) compute ρ^k
(c) compute $x^{k+1} = x^k\text{-}\rho^k d^k$

(d) Has the precess converged? That is $\|x^{k+1}-x^k\| \leq \partial$, $\partial > 0$ for a preassigned ∂.
Yes: Then $x^{k+1} = x^k$
No: Set k = k+1 and go back to a above.

CONVERGENCE THEOREM

Theorem 2: Let $\{\rho^k\}$ be a real sequence satisfying

$$\text{(i) } o < \rho^k < 1 \qquad \forall\, k \geq 1$$

$$\text{(ii) } \sum_{k=0}^{\infty} \rho^{2k} < \infty$$

Then the stochastic sequence $\{x^k\}_{k=1}^{\infty}$ defined by $x^0 \in D(T)$, $x^{k+1} = x^k - \rho^k d^k$ remains in D(t) and converges strongly to the fixed point of T almost surely

Proof

$$\text{Let } q_j = \rho^j \left\| d^j - \frac{\partial f(x^j)}{\partial x} \right\|$$

so that $\{q_j\}$ is a sequence of independent identically distributed random variables with $Eq_j = 0$ for each j.
Thus the sequence of partial sums

$$S_n = \sum_{j=1}^{n} q_j$$

is a martingale. But

$$ES_n^2 = \sum_{j=1}^{n} Eq_j^2$$

$$= \sum \rho^{2j} E \left\| dj - \frac{\partial f(x^j)}{\partial x} \right\|^2$$

$$= M^{-1}\sigma^2 \sum \rho^{2j}$$

Thus by the hypothesis $ES_n^2 < \infty$. Hence by martingale convergence theorem(Whittle,1976), we have $\sum_{n=1}^{\infty} q_j < \infty$. So that

$$\lim_{j\to\infty} q_j = \lim_{j\to\infty} \rho^j \left\| d^j - \frac{\partial f(x^j)}{\partial x} \right\| = 0$$

Hence by Throrem 1, the unique fixed point x^* such that

$$x^* = \varphi(x^*)$$

where

$$\varphi(x^j) = x^j - \rho^j \frac{\partial f(x^j)}{\partial x}$$

satisfying

$$\left\{ x^* : \frac{\partial f(x^*)}{\partial x} = 0 \right\}$$

is limit of the stochastic sequence $x^{j+1} = x^j - \rho^j d^j$ as $j \to \infty$ since $x^j \in D(T)$ $\forall j$ and S is compact, S contains unique fixed point satisfying

$$\left\{ x^* : \frac{\partial f(x^*)}{\partial x} = 0 \right\}$$

Thus, the unique fixed point x^* of T is the limit of every successive approximate

$$x^{j+1} = x^j - \rho^j d^j$$

for an arbitrary non-zero starting $x^o \in S$

REFERENCES

Browder, F.E., 1967. Nonlinear mapping of nonexpansive and accretive type in Banach spaces. Bull. Am. Math. Soc., 73: 875-882.

Chidume, C.E., 1987. Iterative approximation of fixed of point of Lipschitzian strictly Pseudocontractive mappings. Proc. Am. Math. Soc., 99: 283-288.

Gale, D., 1960. The Theory of Linear Economic Models. McGraw-Hill, New York Chapt, 9.

Kemeny, J. and J.L. Snell, 1960. Finite Markov Chains. van Nostiana, Princeton, N.Y.

Pruitt, W.E., 1964. Eigenvalues of non negative matrix. Ann. Math. Stat., 35: 1797-1800.

Solow, R., 1952. On the structure of linear models. Econometrica, 20: 29-49.

Whittle, P., 1976. Probability. John Wiley and Sons Ltd.