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Efficient Estimation of Normal Population Mean

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Abstract: In 1964, Searls provided the Minimum Mean Squared Error (MMSE) estimator $(1 + \sigma^2/n\mu^2)^{-1}\,\bar{x}$ in the class of estimators of the type $k\bar{x}$ for estimating the mean μ of a normal population with variance σ^2. However, as (σ/μ) is seldom known, this MMSE estimator is not very useful, in practice. In 1980, Srivastava, therefore, proposed the correspondingly computable estimator $t = \bar{x}/(1 + s^2/(n\bar{x}^2))$ and showed that it is more efficient than the usual estimator $\bar{x}$ whenever $\sigma^2/(\mu\sigma^2)$ is at least 0.5. Nevertheless, the relevant gain in efficiency would be still unknown as it involves the unknown population parameters μ and σ^2. In 1990, Srivastava and Singh provided an UMVU estimate of the Relative Efficiency ratio, $E(\bar{x}-\mu)^2/E(t-\mu)^2$ to help determine the usefulness of the estimator t over the usual sample mean estimator $\bar{x}$ in practice. In most cases the coefficient of variation of the sample mean estimator $\bar{x}$, which is more stable than the original variable X and hence, its sample counterpart, could be rather low. For such situations, the present study proposes $t^{\circledast} = \bar{x}(1 + s^2/(n\bar{x}^2))$ and studies it on the lines similar to those of the estimator t of Srivastava and Singh. The motivating objective, to improve t in practical situations is amply achieved.

Key words: Normal population mean, estimated coefficient of variation of sample mean, estimated relative efficiency, uniformly minimum variance, unbiased estimate, empirical study

INTRODUCTION

It is well known that the sample mean $\bar{x}$ is the Uniformly Minimum Variance Unbiased (UMVU) estimator of the normal population mean μ. Relaxing the requirement of unbiasedness, Searls (1964) and Gleser and Healy (1976) obtained the Minimum Mean Squared Error (MMSE) and the Minimum Quadratic Risk Scale Invariant estimators, respectively. However, these and other similar estimators are of limited practical use as they depend on the coefficient of variation being known. Very often, such information could not be assumed to be available.

To overcome this difficulty, Srivastava (1980) proposed the following sample counterpart of the MMSE estimator of Searls (1964)

$$t = \bar{x}\Big/\left(1+s^2\Big/\left(n\,\bar{x}^2\right)\right) = \bar{x}\Big/\left(1+\nu\right) \qquad (1)$$

where, $\sqrt{\nu} = s\Big/\left(\bar{x}\sqrt{n}\right)$ is the sample estimate of the coefficient of variation of the sample mean $\bar{x}$. This estimator was subsequently studied by Srivastava and Singh (1990). They determined the UMVU estimate of the relative efficiency ratio of t, with respect to the usual unbiased estimator, $\bar{x}$. They concluded that the estimator t in Eq. 1 is to be recommended over $\bar{x}$ only when $\sqrt{\nu}$ is at least 1.36044. For large values of the sample size n, however, the condition imposed on the estimator might become less stringent. But even for a very large value of n, for the estimator t to be better than $\bar{x}$, $\sqrt{\nu}$ must be >>1.0. It would be rather very common to encounter situations where the coefficient of variation of the sample mean $\bar{x}$, a rather more stable random variable than the original variable study variable X, will not be that large to justify the superior performance of the estimator t. In such situations, we might rather have $\sqrt{\nu} < 1.0$, or even $\sqrt{\nu} << 1.0$. It must be carefully noted that in this context, $\sqrt{\nu} < 1.0$ occurs more often than $\sqrt{\nu} \geq 1.0$. Thus, in most situations $\sqrt{\nu} < 1.0$ or $\sqrt{\nu} << 1.0$. Hence for such, rather frequent cases, we propose an alternative estimator, $t^{\circledast}$, defined by:

$$t^{\circledast} = \bar{x}\left(1+s^2\Big/\left(n\,\bar{x}^2\right)\right) = \bar{x}\left(1+\nu\right) \qquad (2)$$

THE UMVU ESTIMATOR OF THE RELATIVE EFFICIENCY RATIO

Srivastava and Singh (1990) derived the UMVU estimator of the exact Relative Efficiency Ratio of their

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estimator t. The present study does the same for the proposed estimator $t^{\otimes}$, defined in Eq. 2. The relative efficiency of $t^{\otimes}$ with respect to the usual estimator $\bar{x}$ (in %) is

$$\frac{100}{\eta} = 100\frac{E(\bar{x}-\mu)^2}{E(t^{\otimes}-\mu)^2} \qquad (3)$$

Since $(\bar{x}, s^2)$ is a jointly complete sufficient statistic for (μ, σ^2), it suffices to find an unbiased estimator of the Efficiency Ratio, η, as a function of $(\bar{x}, s^2)$ alone. This estimator would be a UMVUE. From (3), we have

$$\eta = \frac{E(t^{\otimes}-\mu)^2}{E(\bar{x}-\mu)^2}$$

$$= \left(\frac{n}{\sigma^2}\right)E\left[(\bar{x}-\mu)+\frac{s^2}{n\bar{x}}\right]^2$$

$$= 1 + 2A + B$$

where $A = (1/\sigma^2)E(s^2(\bar{x} - \mu)/\bar{x})$ and $B = (1/(\sigma^2))E(s^4/(n\bar{x}^2))$. Using the results that $\bar{x} \sim N(\mu, \sigma^2/n)$, $(n - 1)s^2/\sigma^2 \sim \chi^2(n - 1)$ and that $\bar{x}$ and s^2 are independent, unbiased estimators of A and B can be found as follows:

$$A = \left(\frac{1}{\sigma^2}\right)E_{s^2}E_{\bar{x}}\left[\left(\frac{s^2}{\bar{x}}\right)(\bar{x}-\mu)\right]$$

$$= \left(\frac{c}{\sigma^2}\right)E_{s^2}\left[\int_{-\infty}^{+\infty}\left(\frac{s^2}{\bar{x}}\right)(\bar{x}-\mu)\exp\left(\frac{-n(\bar{x}-\mu)^2}{2\sigma^2}\right)d\bar{x}\right],$$

$$c = \left(\frac{n}{2\pi\sigma^2}\right)^{1/2}$$

$$= \left(\frac{-c}{n}\right)E_{s^2}\left[\int_{-\infty}^{+\infty}\left(\frac{s^2}{\bar{x}}\right)\frac{d}{d\bar{x}}\left[\exp\left(\frac{-n(\bar{x}-\mu)^2}{2\sigma^2}\right)\right]d\bar{x}\right]$$

$$= \left(\frac{-c}{n}\right)E_{s^2}\left[\int_{-\infty}^{+\infty}\left(\frac{s^2}{\bar{x}^2}\right)\exp\left(\frac{-n(\bar{x}-\mu)^2}{2\sigma^2}\right)d\bar{x}\right]$$

$$= -\frac{1}{n}E\left(\frac{s^2}{\bar{x}^2}\right) = -E\left(\frac{s^2}{n\bar{x}^2}\right)$$

$$= -E(\nu)$$

Hence $\hat{A} = -\nu$ is an unbiased estimator of A. Further,

$$B = \left(\frac{1}{n\sigma^2}\right)E\left(\frac{s^4}{\bar{x}^2}\right)$$

$$= \left(\frac{1}{n\sigma^2}\right)E_{\bar{x}}E_{s^2}\left(\frac{s^4}{\bar{x}^2}\right)$$

$$= \frac{\sigma^2}{n(n-1)^2}E_{\bar{x}}E_{s^2}\left(\left(\frac{(n-1)s^2}{\sigma^2}\right)^2\frac{1}{\bar{x}^2}\right)$$

$$= \frac{\sigma^2}{n(n-1)^2}E_{\bar{x}}\left(\frac{n^2-1}{\bar{x}^2}\right),$$

$$E\left((n-1)s^2/\sigma^2\right)^2 = E\left(\chi^2(n-1)\right)^2 = n^2-1$$

$$= \frac{n+1}{n-1}E\left(\frac{s^2}{n\bar{x}^2}\right),\ E(s^2) = \sigma^2$$

$$= \frac{n+1}{n-1}E(\nu)$$

Hence $\hat{B} = (n+1)/(n-1)\nu$ is an unbiased estimator of B. Let $\hat{\eta} = 1 + 2\hat{A} + \hat{B}$. Then $\hat{\eta}$ is an unbiased estimator of η and hence an UMVU estimator of μ. We can write $\hat{\eta}$ as

$$\hat{\eta} = 1 - 2\nu + \left(\frac{n+1}{n-1}\right)\nu = 1 - \left(\frac{n-3}{n-1}\right)\nu \qquad (4)$$

EMPIRICAL STUDY AND CONCLUSIONS

The present study shows that $\hat{\eta}$ is an UMVU estimator of the relative efficiency ratio μ, where $\eta = E(t^{\otimes} - \mu)^2/E(\bar{x} - \mu)$. Thus, our proposed estimator $t^{\otimes}$ will be more efficient than the usual estimator $\bar{x}$ if $0<\hat{\eta}<1$, that is, if

$$0 < \left(\frac{n-3}{n-1}\right)\nu < 1 \qquad (5)$$

However, since $0<(n - 3)/(n - 1)<1$ for $n<1$ or $n>3$ and $0<\nu<1$, condition (5) holds for $n>3$.

Since the sample mean $\bar{x}$ is a more stable variable than the original study variable X, its coefficient of variation is most likely less than 1, rather than greater than 1.36, as recommended by Srivastava and Singh (1990), for using the estimator t.

Further, to have an idea of the estimated gain in efficiency by using our proposed estimator $t^{\otimes}$ rather than

Table 1: Relative Efficiency (%) of t^{*} relative to $\bar{x}$

n/v	0.05	0.10	0.20	0.50	0.90
5	102.56	105.26	111.11	133.33	181.82
15	104.48	109.38	120.69	175.00	437.50
30	104.88	110.27	122.88	188.00	617.02
60	105.08	110.69	123.95	193.44	766.23
100	10515	110.86	124.37	196.04	846.15

$\bar{x}$, when $v<1$, we have tabulated in Table 1, the results of an empirical study of the efficiency of t^{*} relative to $\bar{x}$, as defined in equation 3. We have used Eq. 4 for some illustrative values of v and n.

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