Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 6 (10): 2151-2157, 2006
ISSN 1812-5654

Order Statistics from Pareto Distribution

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Abstract: In this study, we derive some recurrence relations of single and product moments of order statistics from Pareto distribution. We estimate the parameters of the distribution using the moment of order statistics. We compute the mean, variance and coefficient of variation of order statistics from Pareto distribution.

Key words: Pareto distribution, order statistics, single moment, product moment, recurrence relations, estimation, coefficient of variation

INTRODUCTION

Pareto distribution has a wide use in economic studies. It has played a major part in investigation of several economic phenomina. Arnald (1983) gives an extensive historical survey of its use in context of income distribution. Khan and Abu ammoh (1999) characterized Pareto distribution by conditional expectation of order statistics. Galambos and Kotz (1978) and Ahmed (1991) have used the concept of left truncated moments to characterized some probability distributions like exponential, Pareto, gamma, negative binomial, beta, binomial and Piossion. Adeyemi (2002) derived some recurrence relations for moments of order statistics from a symmetric general log logistic distribution. Balakrishnan *et al.* (1988) reviewed recurrence relations for product moments of order statistics in case of specific distribution concerning exponential, power function, Pareto, Burr, Rayleigh and logistic distributions. Mohie *et al.* (1996) obtained a general identity for product moments of order statistics in a class of distribution functions, including Pareto, Weibull, exponential, Rayleigh and Burr distributions. In this paper, we derive some recurrence relations of single and product moments of order statistics from Pareto distribution. We estimate the parameters of the distribution by using the moment of the first order statistics and the mean, variance and the coefficient of variation are also computed.

$$f(x) = \alpha\gamma^{\alpha}x^{-\alpha-1} \qquad x \geq \gamma,\ \alpha, \gamma > 0 \tag{1}$$

$$F(x) = 1 - \left(\frac{\gamma}{x}\right)^{\gamma} \qquad x \geq \gamma,\ \alpha, \gamma > 0 \tag{2}$$

From (1) and (2) we have

$$xf(x) = \alpha[1 - F(x)] \tag{3}$$

Probability distribution function of $x_{i:n}$ $(1 \leq i \leq n)$ is given by David (1981).

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{4}$$

Let us denote the single moments $E(x_{i:n}^{m})$ by $\mu_{i:n}^{m}$, $1 \leq i \leq n$ and the product moments $E(x_{i:n}Y_{i:n})$ by $\mu_{i,j:n}$, $1 \leq i < j \leq n$.

RECURRENCE RELATIONS OF SINGLE MOMENTS

Consider (1.4), the expected value of $x_{i:n}$ is given by

$$\mu_{i:n}^{m} = E(x_{i:n}^{m}) = \int_{-\infty}^{\infty} x^{m} f_{i:n}(x)dx$$

From (1) and (2) we have

$$x^{m} = \gamma^{m}[1-F(x)]^{-m/\alpha}$$

$$\mu_{i:n}^{m} = c_{i:n}\gamma^{m}\int_{\gamma}^{\infty} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i-m/\alpha}dx$$

$$\mu_{i:n}^{m} = \frac{\gamma^{m}n!(n-i-m/\alpha)}{(n-i)!(n-m/\alpha)!} \tag{5}$$

Theorem 1: Replace I-1 for I in (5) for $(1 \leq i \leq n-1)$ we have,

$$\mu_{i-1:n}^{m} = \frac{\gamma^{m}n!(n-i-m/\alpha+1)}{(n-i+1)!(n-m/\alpha)!}$$

Then we have

$$\mu_{i:n}^{m} = \left(\frac{n-i+1}{n-i-m/\alpha+1}\right)\mu_{i-1:n}^{m}$$

Theorem 2: For (1 ≤i≤n-1), let i = 1 in (5) we have

$$\mu_{1:n}^{m} = \frac{\gamma^{m} n!(n - m/\alpha - 1)}{(n-1)!(n - m/\alpha)!} \quad (6)$$

Put i = n we have

$$\mu_{n:n}^{m} = \frac{\gamma^{m} n!(-m/\alpha)}{(n - m/\alpha)!} \quad (7)$$

From (6) and (7) we have

$$\mu_{n:n}^{m} = \frac{(n-1)!(-m/\alpha)!}{(n - m/\alpha - 1)!}\mu_{1:n}^{m} \quad (8)$$

Theorem 3: Replace i+1 for i in (5) for (1 ≤i≤n-1) we have,

$$\mu_{i+1:n}^{m} = \frac{\gamma^{m} n!(n - i - m/\alpha - 1)}{(n-i-1)!(n - m/\alpha)!}$$

We have that

$$\mu_{i+1:n}^{m} = \left(\frac{n-i}{n-i-m/\alpha}\right)\mu_{i:n}^{m} \quad (9)$$

And

$$\mu_{i+1:n}^{m} - \mu_{i:n}^{m} = \left(\frac{m/\alpha}{n-i}\right)\mu_{i+1:n}^{m} \quad (10)$$

Theorem 4: From (5) and for m = 1, we have

$$\mu_{i:n-1} = \frac{\gamma(n-1)!(n-i-1/\alpha-1)!}{(n-i-1)!(n-1/\alpha-1)!} \quad (11)$$

From (5) and (11) we obtain

$$\mu_{i:n} = \frac{n(n-i-1/\alpha)}{(n-i)(n-1/\alpha)}\mu_{i:n-1} \quad (12)$$

Also we obtain

$$\mu_{i:n-1} - \mu_{i:n} = \left(\frac{i}{n-1/\alpha}\right)\mu_{i:n-1} \quad (13)$$

Theorem 5: In (5) replace i-1 for i and n-1 for n we obtain

$$\mu_{i-1:n-1}^{m} = \frac{\gamma^{m}(n-1)!(n-i-m/\alpha)!}{(n-i)!(n-m/\alpha-1)!} \quad (14)$$

Dividing (5) by (14) we have

$$\mu_{i:n}^{m} = \frac{n}{n-m/\alpha}\mu_{i-1:n-1}^{m} \quad (15)$$

Theorem 6: For 1< i <n-1

$$\mu_{i:n} + \mu_{i+1:n} = A\mu_{i:n-1}$$

Where

$$A = \frac{-n(i+1/\alpha)}{(n-i)(n-1/\alpha)}$$

Proof: Using (5) we have

$$\mu_{i+1:n} = \frac{\gamma n!(n-i-1/\gamma-1)!}{(n-i)!(n-1/\alpha)!} \quad (16)$$

Adding the last equation to (5) and simplifying, we readily obtain the result (16).

Theorem 7: For 1< i <n-1,

$$\mu_{i+1:n+1}^{m+\alpha} - \mu_{i:n}^{m+\alpha} = \frac{n(1+m/\alpha)}{n-i}\mu_{i:n-1}^{m+\alpha} \quad (17)$$

Proof: From (5), we have

$$\mu_{i:n}^{m+\alpha} = \frac{\gamma^{m+\alpha} n!(n-i-m/\alpha-1)!}{(n-i)!(n-m/\alpha-1)!} \quad (i)$$

$$\mu_{i+1:n+1}^{m+\alpha} = \frac{\gamma^{m+\alpha}(n+1)!(n-i-m/\alpha-1)!}{(n-i)!(n-m/\alpha)!} \quad (ii)$$

Subtracting (i) from (ii) and simplifying we will obtain the result (17).

Theorem 8: For 1< i <n-1

$$(1-\alpha(n-i+2)/2)\mu_{i-1:n-1}^{2} = (\alpha(2-i)/2)\mu_{i-2:n-1}^{2} - (\alpha/2)\mu_{i-1:n-1} \quad (18)$$

Proof:

$$\mu_{i-1:n-1}^{2} = \frac{(n-1)!}{(i-2)!(n-i)!}\int_{\gamma}^{\infty} x^{2} f(x)[F(x)]^{i-2}[1-F(x)]^{n-i}dx$$

Using (3), we may written for 1 = i = n-1

$$-\frac{\alpha n!}{2(i-2)!(n-i)!}\int_{\gamma}^{\infty} x^{2} f(x)[F(x)]^{i-2}[1-F(x)]^{n-i}dx \quad (iii)$$

Integrating the RHS of (iii) by parts, treating x for integrating and the rest of integrand for differentiating, we obtain for 1 ≤i≤n-1, the equation

$$\mu^2_{i-1:n-1} = \frac{\alpha n!}{2(i-2)!(n-i)!}[(n-i+1)\int_\gamma^\infty x^2 f(x)[F(x)]^{i-2}[1-F(x)]^{n-i}dx - (i-2)\int_\gamma^\infty x^2 f(x)[F(x)]^{i-3}[1-F(x)]^{n-i+1}dx]$$

If we split the first integral of the RHS of the last equation into two and combine with the second integral, the last equation may be written as:

$$\mu^2_{i-1:n-1} = \frac{\alpha n n!}{2(i-2)!(n-i)!}\int_\gamma^\infty x^2 f(x)[F(x)]^{i-2}[1-F(x)]^{n-i}dx$$
$$-\frac{\alpha n n!}{2(i-3)!(n-i)!}\int_\gamma^\infty x^2 f(x)[F(x)]^{i-3}[1-F(x)]^{n-i}dx$$
$$-\frac{\alpha n n!}{2(i-2)!(n-i)!}\int_\gamma^\infty x^2 f(x)[F(x)]^{i-2}[1-F(x)]^{n-i}dx$$

This equation, when simplified, yields the relation (18).

Theorem 9: For 1<i<n-1,

$$2\mu^2_{i+1:n+1} = \alpha(n+1)[\mu^2_{i+1:n} - \mu^2_{i:n}] \tag{19}$$

Proof:

$$\mu^2_{i+1:n+1} = \frac{(n+1)!}{i!(n-i)!}\int_\gamma^\infty x^2 f(x)[F(x)]^{i}[1-F(x)]^{n-i}dx$$

Using (3) we obtain

$$\mu^2_{i+1:n+1} = \frac{\alpha(n+1)!}{i!(n-i)!}\int_\gamma^\infty x\ f(x)[F(x)]^{i}[1-F(x)]^{n-i+1}dx$$

Integrating by parts we have

$$\mu^2_{i+1:n+1} = \frac{\alpha(n+1)!}{2i!(n-i-1)!}\int_\gamma^\infty x^2 f(x)[F(x)]^{i}[1-F(x)]^{n-i}dx - \frac{\alpha(n+1)!}{2(i-1)!(n-i)!}\int_\gamma^\infty x^2 f(x)[F(X)]^{i-1}[1-F(x)]^{n-i+1}dx$$

This equation, when simplified, we obtain the relation (19).

Theorem 10: For 1< i <n-1,

$$\mu_{i+1:n+1} = \alpha(n+1)[\mu_{i+1:n+1} - \mu_{i:n}] \tag{20}$$

Proof:

$$\mu_{i+1:n+1} = \frac{(n+1)!}{i!(n-i)!}\int_\gamma^\infty x f(x)[F(x)]^{i}[1-F(x)]^{n-i}dx$$

using Eq. (3), the last equation becomes

$$\mu_{i+1:n+1} = \frac{\alpha(n+1)!}{i!(n-i)!}\int_\gamma^\infty [F(x)]^{i}[1-F(x)]^{n-i+1}dx$$

Integrating by parts treating x, we obtain the result (20).

RECURRENCE RELATIONS OF PRODUCT MOMENTS

The probability density function of $x_{i:n}$ and $y_{j:n}$ $(1\le i<j\le n)$, $\gamma \le x, y \le \infty$ is given by

$$f_{i,j:n}(x,y)=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}f(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}$$

is defined by x^r, y^r. The expected value of

$$\begin{aligned}\mu_{i,j:n}^{r,s}&=E(x^ry^s)\\&=\int_{x=\gamma}^{\infty}\int_{y=x}^{\infty}x^ry^sf_{i,j:n}(x,y)dxdy\\&=\frac{\gamma^{r+s}n!(n-j-s/\alpha)!(n-s/\alpha-r/\alpha-i)!}{(n-j)!(n-s/\alpha-i)!(n-s/\alpha-r/\alpha)!}\end{aligned} \tag{21}$$

We let r = s = 1, in Eq. (21), we obtain the well known results (Johnson and Kotz, 1970).

$$\mu_{i,j:n}=\frac{\gamma^2n!(n-j-1/\alpha)!(n-2/\alpha-i)!}{(n-j)!(n-1/\alpha-i)!(n-2/\alpha)!} \tag{22}$$

Theorem 11: For $1\le i<j\le n$

$$(1-\alpha)\mu_{i,j:n}=\alpha\left[n\mu_{i,j-1:n-1}-\mu_{i,j-1:n}\right] \tag{23}$$

Proof:

$$\mu_{i,j:n}=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\iint xyf(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}$$

$$[1-F(y)]^{n-j}dxdy \tag{24}$$

Using (3) we have

$$\mu_{i,j:n}=\frac{\alpha n!}{(i-1)!(j-i-1)!(n-j)!}\iint xf(x)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j+1}dxdy$$

$$I_y=\int_x^{\infty}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j+1}dy \tag{24a}$$

Integrating (24a) by parts then subsititiute in (24), we have

$$\mu_{i,j:n}=\frac{\alpha n!}{(i-1)!(j-i-1)!(n-j)!}[(n-j-1)\iint xyf(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}$$

$$[1-F(y)]^{n-j}dxdy-(j-i-1)\iint xyf(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-2}$$

$$[1-F(y)]^{n-j+1}dxdy$$

On simplifying the last equation, the result (23) will obtained.

Theorem 12: For $1\le i<j\le n$

$$(s-\alpha)\mu_{i,j:n}^{r,s}=\alpha(n\mu_{i,j-1:n-1}^{r,s}-\mu_{i,j-1:n}^{r,s}) \tag{25}$$

Proof

$$\mu^{r,s}{}_{i,j:n}=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\iint x^ry^sf(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}$$

$$[1-F(y)]^{n-j}dxdy$$

Using (3) we have

$$\mu^{r,s}_{i,j:n} = \frac{\alpha n!}{(i-1)!(j-i-1)!(n-j)!}\iint x^r y^{s-1} f(x)f(y)[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}$$
$$[1-F(y)]^{n-j+1}dxdy \quad (26)$$

Let

$$I_y = \int y^{s-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j+1}dy \quad (26a)$$

Integrating (26a) by parts, then subsititute in equation (26) and simplifying the result (25) will obtained.

Theorem 13: For $1 \le i < j \le n$

$$(n-j)\mu^{r-\alpha,s+\alpha}_{i,j:n} + (i-j)\mu^{r,s}_{i,j+1:n} = n\mu^{r,s-\alpha}_{i,j:n-1} \quad (27)$$

Proof: From (21) we have

$$(n-j)\mu^{r-\alpha,s+\alpha}_{i,j+1:n} = \frac{\gamma^{r+s}n!(n-j-s/\alpha-1)!(n-s/\alpha-r/\alpha-i)!}{(n-j-1)!(n-s/\alpha-i-1)!(n-s/\alpha-r/\alpha)!} \quad (28)$$

From (21) we have

$$(i-j)\mu^{r,s}_{i,j+1:n} = \frac{\gamma^{r+s}n!(i-j)(n-j-s/\alpha-1)!(n-s/\alpha-r/\alpha-i)!}{(n-j-1)!(n-s/\alpha-i)!(n-s/\alpha-r/\alpha)!} \quad (29)$$

Adding (28) to (29) and simplifying we obtain the result (27). Also we obtain that

$$\mu^{r,s}_{i,j:n} = \left(\frac{n-j-s/\alpha-1}{n-j-s/\alpha+1}\right)\mu^{r+\alpha,s-\alpha}_{i,j:n} \quad (30)$$

Theorem 14: For $1 \le i < j \le n$

$$\mu_{i+1,j:n} - \mu_{i,j:n} = \frac{n}{(n\alpha-2)(n-i-2/\alpha)}\mu_{i-1,j-1:n-1} \quad (31)$$

Proof: From (22) we have

$$\mu_{i+1:n} = \frac{\gamma^2 n!(n-j-1/\alpha)!(n-i-2/\alpha-1)!}{(n-j)!(n-i-1/\alpha-1)!(n-2/\alpha)!} \quad (32)$$

Subtracting (22) into (32) and simplifying we obtain the result (31).

Theorem 15: For $1 \le i < j \le n\text{-}1$ for

$$(n+1)[\mu_{i,j+1:n} - \mu_{i,j:n}] = \frac{n\alpha+\alpha-2}{(n\alpha-j\alpha-1)(n\alpha-2)}\mu_{i+1,j+1:n+1} \quad (33)$$

Proof: Replace j+1 with j in (22) we can easily obtain

$$\mu_{i,j+1:n} = \frac{\gamma^2 n!(n-j-1/\alpha-1)!(n-i-2/\alpha)!}{(n-j-1)!(n-i-1/\alpha)!(n-2/\alpha)!} \quad (34)$$

From (22) and (34) we obtain the relation (33).

Using (21), some recurrence relations for product moments of order statistics from Pareto distribution can be obtained by simple rearrangement and manipulation, they are

$$\gamma^{\alpha}\mu^{r,s}_{i,j:n} = \left(\frac{n-s/\alpha-r/\alpha-i}{n-s/\alpha-r/\alpha}\right)\mu^{r+\alpha,s}_{i,j:n} \tag{35}$$

$$\mu^{r+\alpha,s}_{i,j:n} = \left(\frac{n-j-s/\alpha}{n-i-s/\alpha}\right)\mu^{r,s+\alpha}_{i,j:n} \tag{36}$$

$$\mu^{r,s}_{i,j:n} = \frac{n-j-s/\alpha}{n-s}\mu^{r,s}_{i,j+1:n} \tag{37}$$

$$\mu^{r,s}_{i,j:n} = \left(\frac{n-s/\alpha-r/\alpha-i}{n-s/\alpha-i}\right)\mu^{r,s}_{i+1,j:n} \tag{38}$$

$$\gamma^{\alpha}\mu^{r,s}_{i,j:n} = \frac{(n-j-s/\alpha)(n-s/\alpha-r/\alpha-i)}{(n-s/\alpha-i)(n-s/\alpha-r/\alpha)}\mu^{r,s+\alpha}_{i,j:n} \tag{39}$$

$$\mu^{r,s}_{i,j:n} = \frac{(n-j+1)(n-s/\alpha-i+1)}{(n-j-s/\alpha+1)(n-s/\alpha-r/\alpha-i+1)}\mu^{r,s}_{i-1,j-1:n} \tag{40}$$

$$\mu_{i,j:n} = \frac{(n-j-1/\alpha)(n-j-2/\alpha)}{(n-j)(n-i-1/\alpha)}\mu_{i-1,j-1:n} \tag{41}$$

$$\mu_{n,n:n} = \frac{(n-1)!(-2/\alpha)!}{(n-2/\alpha-1)!}\mu_{1,1:n} \tag{42}$$

APPLICATION

a) Put m = 1, in (6) we have

$$\mu_{1:n} = \frac{\alpha\gamma n}{\alpha n-1} \tag{43}$$

The variance of $x_{i:n}$ is given by

$$\mathrm{Var}(x_{1:n}) = \mu^2_{1:n} - [\mu_{1:n}]^2$$

$$= \frac{\alpha\gamma^2 n}{(n\alpha-1)^2(n\alpha-2)} \tag{44}$$

Dividing (44) by (43) we obtain

$$\frac{\mathrm{Var}(x_{1:n})}{\mu_{1:n}} = \frac{1}{(n\alpha-1)^2} \tag{45}$$

Equating the right hand side of (45) to $S^2/X_{1:n}$ we have

$$S^2/X_{1:n} = \frac{1}{(n\alpha-1)^2} \tag{46}$$

Where, $X_{1:n}$ is the smallest value of the sample and S^2 is the variance of the sample. We solved Eq. (46) for α then subsititute in (43) to obtain γ.

We generate a data set from Pareto distribution with parameters α = 1 and γ = 2

And a sample of size 20, using equations (46) and (43) we obtain the estimates for the parameters α and γ as 1 and 1.9997, respectively.

b) From (5) and for m = 1, α = 1, we have

$$\mu_{i;n} = \frac{2n}{n-i} \qquad n>i$$

For m = 2, we have

$$\mu^2_{i:n} = \frac{4n(n-1)}{(n-i)(n-i-1)}, \qquad n>i+1$$

The variance of $X_{i,n}$ is given by

$$Var(x_{i:n}) = \mu^2_{i:n} - [\mu_{i:n}]^2$$

The coefficient of variation (CV) is given by

$$CV = \frac{\sqrt{Var(x_{i:n})}}{\mu_{i:n}}$$

We compute the mean, variance and coefficient of variation of order statistics from Pareto distribution up to 10. The results are shown in Table 1-3.

Table 1: The mean of order statistics from pareto distribution

n/i	1	2	3	4	5	6	7	8	9	10
1										
2	4.00									
3	3.00	3.00								
4	2.67	4.00	8.00							
5	2.50	3.33	5.00	10.00						
6	2.40	3.00	4.00	6.00	1.20					
7	2.33	2.80	3.50	4.67	7.00	14.00				
8	2.29	2.67	3.20	4.00	5.33	8.00	16.00			
9	2.25	2.57	3.00	3.60	4.50	6.00	9.00	18.00		
10	2.22	2.50	2.86	3.33	4.00	5.00	6.67	10.00	20.00	

Table 2: The variance of the order statistics from pareto distribution

n/i	1	2	3	4	5	6	7	8	9	10
1										
2										
3	3.00									
4	0.87	8.00								
5	3.75	2.24	15.00							
6	0.24	1.00	4.00	24.00						
7	0.17	0.56	1.75	20.19	35.00					
8	1.16	0.34	0.96	2.67	8.92	48.00				
9	0.08	0.26	0.60	1.44	3.75	12.00	63.00			
10	0.072	0.18	0.39	0.91	2.00	5.00	15.51	80.00		

Table 3: CV of the order statistics from pareto distribution

n/i	1	2	3	4	5	6	7	8	9	10
1										
2										
3	0.577									
4	0.350	0.707								
5	0.775	0.450	0.775							
6	0.200	0.330	0.50	0.816						
7	0.177	0.270	0.38	0.960	0.8845					
8	0.470	0.220	0.31	0.410	0.56	0.74				
9	0.126	0.198	0.26	0.330	0.43	0.58	0.77			
10	0.121	0.170	0.22	0.290	0.35	0.45	0.59	0.89		

REFERENCES

Adeyemi, S., 2002. Some recurrence relations for moments of order statistics from a symmetric generalized log logistic distribution. Interstat. Dec.

Ahmed, A., 1991. Characterization of beta, binomial and Poisson distributions. IEEE Trans. Reliability, 40: 290-295.

Arnold, B.C., 1983. The Pareto distribution. International Cooperative Pupllishing House, Fairland, Maryland.

Balakrishnan, N., H.J. Malik and S.E. Ahmed, 1988. Recurrence relations and identities for moments of order statistics II: Specific continuous distribution. Commmun. Statist. Theor-Meth., 17: 2657-2694.

David, H.A., 1981. Order statistics. 2nd Edn., John Wiely, New York.

Galambos, J. and S. Kotz, 1978. Characterization of probability distributions. Springer Verlag.

Khan, A.H. and A.M. Abuammoh, 1999. Characterization of distributions by conditional expectation of order statistics. J. Applied Sci., 9: 159-168.

Mohie, El Din, M.M. Mahmoud and S.E. Abu-Youssef, 1996. An identity for the product moments of order statistics. Metrika, 44: 95-100.